Normal zones on Zeeman manifolds with trace class heat and Feynman kernels and well defined zonal Feynman integrals

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Abstract

The problem of infinities (divergent integrals) appearing in quantum field theory are treated by *renormalization* in the current theories. By this perturbative tool the desired finite quantities are produced by differences of infinities. The most common reason for these infinities appear is the infinite trace of kernels such as the Wiener-Kac kernel e^{-tH}, or, the Dirac-Feynman kernel $e^{-tH\mathbf{i}}$, thus, they assign infinite measures to physical objects such as self-mass, self-charge, e.t.c..

This paper gives a new non-perturbative approach to this problem. Namely, the Hilbert space $\mathcal{H}$, on which the quantum Hamilton operator H is acting, is decomposed into H-invariant Zeeman zones on which both the Wiener-Kac and the Dirac-Feynman kernels become of the trace class, both defining the corresponding zonal measures on the path-spaces rigorously.

The Hamilton operators, H, considered in this paper are those corresponding to electrons orbiting in a constant magnetic field. This is one of the most important Hamiltonians, introduced for explaining the Zeeman effect. This Hamiltonian is identified with the Laplacian of of certain Riemannian, called Zeeman manifolds. The "particles" modeled by them are called *Zeeman particles*. The spectral Zeeman zone decomposition is introduced, by one of the definitions, by the spectrum of the magnetic dipole moment operator. A zone corresponds to a magnetic state of these particles.

All the important spectral theoretical objects are explicitly established. They include the spectrum, the zonal projection operators, the zonal Wiener-Kac and Dirac-Feynman kernels, and the zonal partition functions.

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1 Introduction

Zones on Zeeman manifolds

Infinities in Quantum Electrodynamics (QED). The problem of infinities (divergent integrals), which is present in calculations since the early days of quantum field theory (Heisenberg-Pauli (1929-30)) or elementary particle physics (Oppenheimer (1930) and Waller (1930) in electron theory) is treated by *renormalization* in the current theories. This perturbative tool provides the desired finite quantities by differences of infinities. This problem originated from concepts such as *point mass* and *point charge* of classical electron theory, which provided the first warning that a point electron will have infinite electromagnetic self-mass: the mass $e^2/6\pi ac^2$ for a surface distribution of charge with radius a blows up for $a \to 0$. In quantum field theory the Hamiltonian of the field is proportional to this electromagnetic self-mass. This is why this infinity launched one of the deepest crisis-es in the history of physics.

The infinities mostly appear in the form of infinite trace of kernels such as the Wiener-Kac kernel e^{-tH}, or, the Dirac-Feynman kernel $e^{-tH\mathbf{i}}$. The WK-kernel provides the fundamental solution of the heat equation while the DF-kernel provides the fundamental solution of the Schrödinger equation. The infinite trace assigns infinite measures to physical objects such as self-mass, self-charge, e.t.c.. Because of the divergent integrals emerging in its construction, also the Feynman measure, which is analogous to the well defined Wiener-Kac measure on the path-spaces, requires renormalization.

This paper offers a new non-perturbative approach to this problem. The main idea in this approach can be briefly described as follows. In the first step the quantum Hilbert space $\mathcal{H}$ (on which the quantum Hamilton operator H is acting) is decomposed into the direct sum of H-invariant subspaces, called Zeeman zones. Then all the operator-actions, such as the the heat- or Feynman-flows, are considered on these invariant subspaces separately. It turns out that both the Wiener-Kac and Dirac-Feynman kernels are of the trace class on each zone, furthermore, both define the corresponding zonal measures on the path-spaces rigorously.

The Hamilton operator considered in this paper is the classical Zeeman operator

$$H_Z = -\frac{\hbar^2}{2\mu}\Delta_{(x,y)} - \frac{\hbar eB}{2\mu c\mathbf{i}}D_z \bullet + \frac{e^2B^2}{8\mu c^2}(x^2+y^2), \tag{1}$$

of a free charged particle. At the time it was introduced, the new feature of

this operator was the orbital angular momentum operator $D_z\bullet = x\partial_y - y\partial_x$, which was the forerunner for an adequate spin-concept. This operator is the result of a long agonizing creative effort [To], searching for a Hamilton operator explaining the Zeeman effect. (Note that the $D_z\bullet$ commutes with the rest part, $\mathbf{O}$, of the complete operator. Thus the spectrum appears on common eigenfunctions, resulting that the $D_z\bullet$ splits the spectral lines of $\mathbf{O}$ (Zeeman effect).) Pauli, who added a spin angular momentum operator to the orbital one, developed the non-relativistic spin-concept. The relativistic concept due to Dirac. Actually, the H_Z is the Hamilton operator of an electron orbiting about the origin of the (x, y)-plane in a constant magnetic field $\mathbf{K} = B\partial_z$. It had been established by means of the Maxwell equations.

Mathematical modeling; Zeeman manifolds. An interesting feature of Zeeman operators, H_Z, is that they can be identified with the Laplace operators on certain Riemannian manifolds, namely, with the Laplacians on two step nilpotent Lie groups endowed with the natural left invariant metrics. The details are as follows.

A 2-step nilpotent metric Lie group is defined on the product $\mathbf{v} \oplus \mathbf{z}$ of Euclidean spaces, $\mathbf{v} = \mathbf{R}^k$ and $\mathbf{z} = \mathbf{R}^l$, where the components are called X- and Z-space respectively. The main object defining the Lie algebra is the linear space $J_{\mathbf{z}}$ of skew endomorphisms J_Z, $Z \in \mathbf{z}$, acting on the X-space. The metric, g, is the left invariant extension of the natural Euclidean metric on the Lie algebra. Particular 2-step nilpotent Lie groups are the Heisenberg-type Lie groups which are defined by endomorphism spaces satisfying the Clifford condition $J_Z^2 = -|Z|^2 id$. They are attached to Clifford modules. Each group, (N, g), extends into a solvable group (SN, g_s).

In this introduction just the relatively simple H-type groups will be considered. On them the Laplacian appears in the form

$$\Delta = \Delta_X + (1 + \frac{1}{4}|X|^2)\Delta_Z + \sum_{\alpha=1}^{l} \partial_\alpha D_\alpha \bullet, \tag{2}$$

where ∂_α is partial derivative on the Z-space $\mathbf{z} = \mathbf{R}^l$ and $D_\alpha\bullet$ denotes the directional derivative along the X-field $J_\alpha(X)$ defined for the natural basis element e_α in the Z-space. Center periodic H-type groups are introduced by factorizations, $\Gamma_\gamma \backslash H$, with Z-lattices $\Gamma_\gamma = \{Z_\gamma\}$ defined on the Z-space. On these manifolds the Laplacian appears in a much more handy form. In fact, in this case the L^2 function space is the direct sum of function spaces W_γ spanned by functions of the form $\Psi_\gamma(X, Z) = \psi(X)e^{2\pi \mathbf{i}\langle Z_\gamma, Z\rangle}$. Each W_γ is invariant under the action of the Laplacian, i. e., $\Delta\Psi_\gamma(X, Z) =$

$\Box_\gamma \psi(X) e^{2\pi \mathbf{i}\langle Z_\gamma, Z\rangle}$, where operator $\Box_\gamma$, acting on $L^2(\mathbf{v})$, is of the form

$$\Box_\gamma = \Delta_X + 2\pi \mathbf{i} D_\gamma \bullet -4\pi^2 |Z_\gamma|^2 (1 + \frac{1}{4}|X|^2). \tag{3}$$

Notice that (1) is nothing but (3) on the 3D-Heisenberg group. On a $(k+1)$-dimensional Heisenberg group, defined by a complex structure J acting on the even dimensional Euclidean space $\mathbf{v} = \mathbf{R}^k$, the Laplacian (3) appears in the form

$$\Box_\lambda = \Delta_X + 2\mathbf{i} D_\lambda \bullet -\lambda^2 |X|^2 - 4\lambda^2. \tag{4}$$

Number $k/2$ is interpreted as the number of particles. The single complex structure J is interpreted such that these particles are identical, rotating in the same plane defined by the same constant magnetic field B.

Operators (1) and (4) are identified by $H = -(1/2)\Box_\lambda$ and by the particular choice $\mu = \hbar = 1, \lambda = eB/2c$ of the constants. Operator (3) contains also the constant $-4\pi^2|Z_\alpha|^2 = 4\lambda^2$, which is proportional to $\hbar^2$ on the microscopic level, thus, it is usually neglected in quantum physics. Also note that the particles described by these Hamiltonians are free ($V = 0$).

The Zeeman operator appears as Laplacian on center periodic 2-step nilpotent Lie groups in a more complex form. These models represent $k/2$ number of charged particles, each of them is orbiting in its own constant magnetic field. The system can be in crystal states represented by the endomorphisms J_γ. The Hamilton operators belonging to these crystal states are $-\frac{1}{2}\Box_\gamma$. This model matches Dirac's famous multi-time theory.

The Riemannian manifolds introduced so far are prototypes of a general Zeeman manifold concept. This general concept is beyond the scope of this article and will be developed in a subsequent paper. This exposition proceeds with considering 2-step nilpotent Lie groups.

Introducing the zones. In [Sz5, Sz6] the problem of infinities is approached by the above mentioned *Zeeman zone decomposition* of the Hilbert space of complex valued L^2-functions defined on the X-space. This *Hilbert space*, $\mathcal{H}$, is isomorphic to the weighted space defined by the Gauss density $d\eta_\lambda(X) = e^{-\lambda|X|^2} dX$. This Hilbert space is spanned by the complex valued polynomials.

Next the Hilbert space is interpreted in this way. The natural *complex Heisenberg group representation* on $\mathcal{H}$ is defined by

$$\rho_{\mathbf{c}}(z_i)(\psi) = (-\partial_{\overline{z}_i} + \lambda_i z_i \cdot)\psi \quad , \quad \rho_{\mathbf{c}}(\overline{z}_i)(\psi) = \partial_{z_i}\psi, \tag{5}$$

where $\{z_i\}$ is a complex coordinate system on the X-space. This representation is reducible. In fact, it is irreducible on the space of holomorphic functions, where it is called Fock representation. Besides the holomorphic subspace there are infinitely many other irreducible invariant subspaces. In the literature only the Fock representation, defined on the space of holomorphic functions, is well known. The above representation is called *extended Fock representation*. In the function operator correspondence, this representation associates operator (1) to the Hamilton function of an electron orbiting in a constant magnetic field.

The zones are defined in two different ways. First, they can be defined by the invariant subspaces of representation (5). The actual construction uses Gram-Schmidt orthogonalization. On the complex plane $\mathbf{v} = \mathbf{C}$, which corresponds to the 2D-Zeeman operator (1), the first zone, $\mathcal{H}^{(0)}$, is the holomorphic zone spanned by the holomorphic polynomials. To construct the second zone one considers the function space $G^{(1)}$ consisting of functions of the form $\overline{z}h$, where the h is a holomorphic function. Then $\mathcal{H}^{(1)}$ is the orthogonal component of $G^{(1)}$ to the previous zone. E. t. c., one can construct all the zones, $\mathcal{H}^{(a)}_{\lambda}$, by continuing the Gram-Schmidt orthogonalization applied to function spaces $G^{(a)}$ spanned by functions of the form $\overline{z}^a h$. The zone index a indicates the maximal number of antiholomorphic coordinates $\overline{z}_i$ in the polynomials spanning the zone.

In the 2D-case the zones are irreducible under the action of the extended Fock representation. In the higher dimensional cases the Gram-Schmidt process results reducible zones, called *gross zones.* More precisely, the holomorphic zone is always irreducible and the gross zones of higher indexes decompose into *irreducible zones*, which can also be explicitly described.

The second technique defines the very same zones by computing the spectrum and the corresponding eigenfunctions explicitly. According to these computations, the eigenfunctions appear in the form

$$h^{(p,\upsilon)}(X) = H^{(p,\upsilon)}(X)e^{-\lambda|X|^2/2} \tag{6}$$

with the corresponding eigenvalues

$$-((4p+k)\lambda + 4k\lambda^2), \tag{7}$$

where p resp. υ are the holomorphic resp. antiholomorphic degrees of the polynomial $H^{(p,\upsilon)}$. Numbers $l = p + \upsilon$ and $m = 2p - l$ are called azimuthal and magnetic quantum numbers respectively. The above function is an eigenfunction also of the magnetic dipole moment operator with eigenvalue m.

Now the zones are created such that the above eigenfunction falls into the zone with the zone index υ. According to the formula $\upsilon = \frac{1}{2}(l-m)$, the zones are determined by the magnetic quantum number m. Thus a zone exhibits the magnetic state of a zonal particle.

Note that the eigenvalues (7) are independent of the antiholomorphic index and they depend just on the holomorphic index. As a result, each eigenvalue has infinite multiplicities. On the irreducible zones, however, each multiplicity is $k/2$. (Here we suppose that there is only one parameter λ involved, meaning that the particles are identical. If the particles (i. e., the λ_i's) are properly distinct, then the multiplicity is 1 on each zone.) Moreover, two irreducible zones are isospectral.

Normal de Broglie Geometry

Introducing the point-spreads by projection kernels. It is remarkable that all the important objects such as the projections onto the zones, the zonal heat- and Feynman-kernels and their well defined trace, and the several Feynman-Kac type formulas can explicitly be computed [Sz5, Sz6].

First the *projection operators*, $\delta^{(a)}$, onto the zones $\mathcal{H}^{(a)}$ are established. If $\{\varphi_i^{(a)}\}_{i=1}^{\infty}$ is an orthonormal basis in $\mathcal{H}^{(a)}$, then the corresponding projection can be formally defined as convolution with the kernel

$$\delta^{(a)}(z,w) = \sum_i \varphi_i^{(a)}(z)\overline{\varphi_i^{(a)}(w)}, \tag{8}$$

where z and w represent complex vectors on $\mathbf{C}^{\frac{k}{2}} = \mathbf{R}^k$. Interestingly enough, these self-adjoint operators are integral operators having smooth Hermitian integral kernels. These kernels can be interpreted as restrictions of the global Dirac delta distribution, $\delta_z(w) = \sum \varphi_i(z)\overline{\varphi}_i(w)$, onto the zones. They have the following explicit form

$$\delta_{\lambda z}^{(a)}(w) = \frac{\lambda^{k/2}}{\pi^{k/2}} L_a^{((k/2)-1)}(\lambda|z-w|^2) e^{\lambda(z\cdot\overline{w} - \frac{1}{2}(|z|^2+|w|^2)}, \tag{9}$$

where $L_a^{((k/2)-1)}(t)$ is the corresponding Laguerre polynomial. Among these kernels only the first one, the projection kernel onto the holomorphic zone, is well known. It is nothing but the Bergman kernel. The new mathematical feature of these formulas is that they are explicitly determined regarding each zone and not just for the holomorphic zone.

These kernels represent one of the most important concepts in this theory. They can be interpreted such that, on a zone, a point particle appears

as a spread described by the above wave-kernel. Note that how these kernels, called zonal point-spread, are derived from the one defined for the holomorphic zone. This holomorphic spread is just multiplied by the radial Laguerre polynomial corresponding to the zone. These point-spreads show the most definite similarity to the de Broglie waves packets (cf. [Bo], pages 61). These zonal kernels can be interpreted such that a point particle concentrated at a point Z appears on the zone as an object which spreads around Z as a wave-package with wave-function described by the above kernel explicitly.

The wave-package interpretation of physical objects started out with the de Broglie theory. This concept was finalized in the Schrödinger equation. The mathematical formalism did not follow this development, however, and the Schrödinger theory is built up on such mathematical background which does not exclude the existence of the controversial point objects. On the contrary, an electron must be considered as a point-object in the Schrödinger theory as well (cf. Weisskopf's argument on this problem in [Schw, Sz5]). An other demonstration for the presence of point particles in the classical theory is the duality principle, stating that objects manifest themselves sometime as waves and sometime as point particles. The bridge between the two visualizations is built up in Born's probabilistic theory, where the probability for that a particle, attached to a wave ξ, can be found on a domain D is measured by $\int_D \xi\overline{\xi}$.

These controversial point-objects, by having infinite self-mass or self-charge attributed to them by the Schrödinger equation, launched one of the deepest crisis's in the history of physics. In the zonal theory de Broglie's idea is established on a mathematical level. Although the points are ostracized from this theory, the point-spreads still bear some reminiscence of the point-particles. For instance, they are the most compressed wave-packages and all the other wave-functions in the zone can be expressed as a unique superposition of the point-spreads. If ξ is a zone-function, the above integral measures the probability of that that the center of a point-spread is on the domain D. This interpretation restores, in some extend, the duality principle in the zonal theory.

Function $\delta^{(a)}_{\lambda Z}\overline{\delta}^{(a)}_{\lambda Z}$ is called the density of the spread around Z. By this reason, function $\delta^{(a)}_{\lambda Z}$ is called spread-amplitude. Both the spread-amplitude and spread-density generate well defined measures on the path-space consisting of continuous curves connecting two arbitrary points. Both measures can be constructed by the method applied in constructing the Wiener measure.

The point-spread concept bears some remote reminiscence of Heisen-

berg's suggestion (1938) for the existence of a fundamental length L, analogously to h, such that field theory was valid only for distances larger than L and so divergent integrals would be cut off at that distance. This idea has never became an effective theory, however. Other distant relatives of the point-spread concept are the smeared operators, i. e. those suitably averaged over small regions of space-time, considered by Bohr and Rosenfeld in quantum field theory. There are also other theories where an electron is considered to be extended. Most of them fail on lacking the explanation for the question: Why does an extended electron not blow up? The zonal theory is checked against this problem in [Sz5], section (F) "Linking to the blackbody radiation; Solid zonal particles".

Global Wiener-Kac and Feynman flows. Both definitions imply that the zones are invariant under the action of the Hamilton operator, therefore, kernels such as the *heat (Wiener-Kac) and Feynman kernels* can be restricted onto them. The zonal kernels are defined by these restrictions.

Since the spectrum is discrete, also the global kernels, defined for the total space $\mathcal{H}$, can be introduced by the trace formula using an orthonormal basis consisting of eigenfunctions on the whole space $\mathcal{H}$. Despite of the infinite multiplicities on the global setting, both global kernels are well defined smooth functions. If the Zeeman operator H_Z is non-degenerated and the distinct non-zero parameters $\{\lambda_i\}$, $i = 1, \dots, r$, are defined on k_i-dimensional subspaces, then for the Wiener-Kac kernel we have

$$d_{1\gamma}(t, X, Y) = e^{-tH_Z}(t, X, Y) = \tag{10}$$
$$= \prod \left(\frac{\lambda_i}{2\pi sinh(\lambda_i t)}\right)^{k_i/2} e^{-\sum \lambda_i (\frac{1}{2} coth(\lambda_i t)|X_i - Y_i|^2 + \mathbf{i}\langle X_i, J(Y_i)\rangle}.$$

This kernel satisfies the Chapman-Kolmogorov identity as well as the limit property $\lim_{t\to 0_+} d_1(t, X, Y) = \delta(X, Y)$, however, it is not of the trace class. Thus functions such as the partition function or the zeta function can not be defined in the standard way. Note that by regularization (renormalization) only well defined relative(!) partition and zeta functions are introduced..

The explicit form of the global Feynman-Dirac kernel is

$$d_{\mathbf{i}}(t, X, Y) = e^{-t\mathbf{i}H_Z}(t, X, Y) = \tag{11}$$
$$= \prod \left(\frac{\lambda_i}{2\pi \mathbf{i} sin(\lambda_i t)}\right)^{k_i/2} e^{\mathbf{i}\sum \lambda_i \{\frac{1}{2} cot(\lambda_i t)|X_i - Y_i|^2 - \langle X_i, J(Y_i)\rangle\}}.$$

Since for fixed t and X function $d_{\mathbf{i}}(t, X, Y)$ is neither L^1- nor L^2-function of variable Y, the integral required for the Chapman-Kolmogorov identity is

not defined for this kernel. It is not of the trace class either. Nevertheless, it satisfies the above limit property. Thus the constructions with the global Feynman-Dirac kernel lead to divergent integrals in the very first step.

It is well known in the history that Kac, who tried to understand Feynman, was able to introduce a rigorously defined measure on the path-spaces only by the kernel e^{-tH}. This measure was, actually, established earlier by Wiener for the Euclidean Laplacian Δ_X. Note that the heat kernel involves a Gauss density which makes this constructions possible. Whereas, the Feynman kernel does not involve such term. This is why no well defined constructions can be carried out with this kernel. One can strait out all this difficulties, however, by considering these constructions on the zones separately.

Zonal Wiener-Kac and Feynman flows. Also the zonal WK- resp. FD-kernels are well defined smooth functions. The gross zonal Wiener-Kac kernels are of the trace class, which can be described, along with their partition functions, by the following explicit formulas.

$$d_1^{(0)}(t,X,Y)=\prod(\frac{\lambda_i e^{-\lambda_i t}}{\pi})^{\frac{k_i}{2}}e^{\sum\lambda_i(-\frac{1}{2}(|X_i|^2+|Y_i|^2)+e^{-2\lambda_i t}\langle X_i,Y_i+\mathbf{i}J(Y_i)\rangle)}, \quad (12)$$

$$d_1^{(a)}(t,X,Y)=(L_a^{(\frac{k}{2}-1)}(\sum\lambda_i|X_i-Y_i|^2)+LT_1^{(a)}(t,X,Y))d_1^{(0)}(t,X,Y), \quad (13)$$

where $LT_1^{(1)}$ is of the form

$$LT_1^{(1)}(t,X,Y))=(1-e^{-2t})lt_1^{(1)}(t,X,Y))= \quad (14)$$
$$(1-e^{-2t})(|X|^2+|Y|^2-1-(1+e^{-2t})\langle X,Y+\mathbf{i}J(Y)\rangle)$$

and for the general terms, $LT_1^{(a)}$, recursion formula can be established. Furthermore,

$$\mathcal{Z}_1^{(a)}(t)=Trd_1^{(a)}(t)=\binom{a+(k/2)-1}{a}\prod\frac{e^{-\frac{k_i\lambda_i t}{2}}}{(1-e^{-2\lambda_i t})^{\frac{k_i}{2}}}=TrD_1^{(a)}(t), \quad (15)$$

where $D_1^{(a)}(t,X,Y)=L_a^{(\frac{k}{2}-1)}(\sum\lambda_i|X_i-Y_i|^2)d_1^{(0)}(t,X,Y)$ is the dominant zonal kernel. The remaining long term kernel in the WK-kernel vanishes for $\lim_{t\to 0_+}$ and is of the 0 trace class. The zonal WK-kernels satisfy the Chapman-Kolmogorov identity along with the limit property $\lim_{t\to 0_+}d_1^{(a)}=\delta^{(a)}$.

Similar statements can be established regarding the zonal DF-flow. The gross zonal Dirac-Feynman kernels are of the trace class which, together

with their partition functions, can be described by the following explicit formulas.

$$d_{\mathbf{i}}^{(0)}(t,X,Y)=\prod(\frac{\lambda_i e^{-\lambda_i t\mathbf{i}}}{\pi})^{\frac{k_i}{2}}e^{\sum\lambda_i(-\frac{1}{2}(|X_i|^2+|Y_i|^2)+e^{-2\lambda_i t\mathbf{i}}\langle X_i,Y_i+\mathbf{i}J(Y_i)\rangle)}, \quad (16)$$

$$d_1^{(a)}(t,X,Y)=(L_a^{(\frac{k}{2}-1)}(\sum\lambda_i|X_i-Y_i|^2)+LT_{\mathbf{i}}^{(1)}(t,X,Y))d_1^{(0)}(t,X,Y), \quad (17)$$

where $LT_{\mathbf{i}}^{(1)}$ is described by

$$LT_{\mathbf{i}}^{(1)}(t,X,Y))=(1-e^{-2t\mathbf{i}})lt_{\mathbf{i}}^{(1)}(t,X,Y))= \quad (18)$$
$$(1-e^{-2t\mathbf{i}})(|X|^2+|Y|^2-1-(1+e^{-2t\mathbf{i}})\langle X,Y+\mathbf{i}J(Y)\rangle)$$

and a general long term, $LT_{\mathbf{i}}^{(a)}$, can be defined recursively. Furthermore,

$$\mathcal{Z}_{\mathbf{i}}^{(a)}(t)=Trd_{\mathbf{i}}^{(a)}(t)=\binom{a+(k/2)-1}{a}\prod\frac{e^{-\frac{k_i\lambda_i t\mathbf{i}}{2}}}{(1-e^{-2\lambda_i t\mathbf{i}})^{\frac{k_i}{2}}}=TrD_{\mathbf{i}}^{(a)}(t), \quad (19)$$

where $D_{\mathbf{i}}^{(a)}(t,X,Y)=L_a^{(\frac{k}{2}-1)}(\sum\lambda_i|X_i-Y_i|^2)d_{\mathbf{i}}^{(0)}(t,X,Y)$ is the dominant kernel. The remaining longterm term in the zonal DF-kernel is of the 0 trace class.

The zonal DF-kernels are zonal fundamental solutions of the Schrödinger equation $(\partial_t+\mathbf{i}(H_Z)_X)d_{\mathbf{i}\gamma}^{(a)}(t,X,Y)=0$, satisfying the Chapman-Kolmogorov identity as well as the limit property $\lim_{t\to 0_+}d_{\mathbf{i}}^{(a)}=\delta^{(a)}$.

On each zone, both the WK- and the FD-kernels are of the trace class, moreover, both define rigorous complex zonal measures, the zonal Wiener-Kac measure $dw_{1xy}^{T(a)}(\omega)$ and the zonal Feynman measure $dw_{\mathbf{i}xy}^{T(a)}(\omega)$, on the space of continuous curves $\omega:[0,T]\to\mathbf{R}^k$ connecting two points x and y. The existence of zonal WK-measure is not surprising, since this measure exists even for the global setting. However, the trace class property is a new feature indeed. In case of the zonal Feynman measure both the trace class property and the existence are new features. Note that the zonal DF-kernels involve a Gauss density which makes these constructions well defined.

PART ONE

ZONES ON ZEEMAN MANIFOLDS

2 Zeeman operators

Zeeman operators defined physically. There is a formal correspondence between Quantum Theory and Classical Theory, formulated by Bohr (1923) as Correspondence Principle. There are many aspects of this complex principle. In short, Quantum Theory substitutes the continuous functions describing the physical systems in Classical Physics by operators acting on a complex Hilbert space $\mathcal{H}$. The discontinuities experienced on the quantum (microscopic) level appear in the discrete spectrum of these operators. This replacement, called *function-operator correspondence*, actually means derivation of the operators from the functions of the classical theory. For instance, the Hamilton operators are derived from the Hamilton functions describing the total energy of the physical systems. The elaboration of a correct correspondence is guided by the requirement (principle): Quantum Theory must approach Classical Theory asymptotically in the limit of large quantum numbers.

It is obvious that the function-operator correspondence should be defined, on the first place, for the canonical coordinates $(q_i, p_i = \dot{q}_i)$. In a time independent physical system the other functions depend only on these coordinates, therefore, one should just extend this correspondence to the other functions, such as the Hamilton functions, by methods known both in physics and geometric quantization. (With some caution, the Taylor expansion is a natural tool for this extension.)

The Heisenberg Lie algebras can be defined by restricting the Poisson brackets

$$\{f, g\} = \sum \partial_{q_i}(f)\partial_{p_i}(g) - \partial_{p_i}(f)\partial_{q_i}(g) \tag{20}$$

onto the linear space of functions spanned by the functions $\{q_i, p_i, 1\}$. The exponential map maps this algebra to the Heisenberg group.

In the quantization process one considers a unitary representation of the Heisenberg group on a complex Hilbert space. In case of the *real Heisenberg groups*, this Hilbert space, $\mathcal{H}$, is the L^2-space of complex valued functions depending on the position coordinates q_i only. I. e., $\mathcal{H} = L^2_{\mathbf{C}}(\mathbf{v}_q)$, where, on the linear space $\mathbf{v}_q$ of position vectors, the complex inner product is defined

by $\langle f,g\rangle = \int f\overline{g}dq$. *Representation* $\rho : f \to f_\rho$ *of the real Heisenberg group* is defined by

$$\rho(q_i)(\psi) = q_i\psi \quad , \quad \rho(p_i)(\psi) = \frac{\hbar}{\mathbf{i}}\partial_{q_i}\psi \quad , \quad \rho(1) = id. \tag{21}$$

The associated operators, $\tilde{f} = \rho(f)$, satisfy the Heisenberg commutative relations:

$$[\tilde{q_i},\tilde{q_j}] = 0 \quad , \quad [\tilde{q_i},\tilde{p_j}] = \hbar\mathbf{i}\delta_{ij} \quad , \quad [\tilde{p_i},\tilde{p_j}] = 0, \tag{22}$$

therefore, the ρ is a Lie algebra representation and the exponential map $e^{\mathbf{i}\rho}$ defines a unitary representation of the Heisenberg group. Representation ρ is called unitary also on the Lie algebra level.

It is well known that this representation is irreducible. By the classical Neumann-Stone theorem the irreducible unitary representations of Heisenberg's groups on a complex infinite dimensional Hilbert space are unique up-to multiplications with unit complex numbers.

The *Zeeman-Hamilton operator* was the result of a longstanding creative effort, searching for a Hamilton operator explaining the Zeeman effect. The exact form of the Hamiltonians was established by considering an electron revolving in a magnetic field $\mathbf{K}$. The classical Hamilton function of this system can be determined by the Maxwell equations

$$\mathbf{K} = \nabla \times \mathbf{a} \quad , \quad \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{a}}{\partial t} - \nabla\phi, \tag{23}$$

describing the electromagnetic field $(\mathbf{E},\mathbf{K})$ in terms of the vector-, $\mathbf{a}$, and scalar-potential, ϕ.

According to these equations, the complete classical Hamilton function of a charged particle in an electromagnetic field that is specified externally (produced by charges and currents other then the one considered)

$$H = \frac{1}{2M}|\mathbf{p} - \frac{Q}{c}\mathbf{a}|^2 + Q\phi + V + \frac{Q^2}{16Mc^2\pi}\int(\mathbf{E}^2 + \mathbf{K}^2)dq, \tag{24}$$

where M is the mass, $\mathbf{p}$ is the kinetic moment, Q is the charge (for an electron $Q = -e$), V is that part of the potential energy which is of non-electromagnetic origin, and the last term is the electromagnetic energy of the field in which the electron is staying. By (21), the Hamilton operator corresponded to this Hamilton function is:

$$\begin{aligned}\rho(H) = -\frac{\hbar^2}{2M}\Delta - \frac{\hbar e}{2Mc}\mathbf{i}(\mathbf{a}\cdot\nabla + \nabla\cdot\mathbf{a}) + \frac{e^2}{2Mc^2}|\mathbf{a}|^2 + \\ +e\phi + V + \frac{e^2}{16Mc^2\pi}\int(\mathbf{E}^2 + \mathbf{K}^2)dq.\end{aligned} \tag{25}$$

In case of the Zeeman-Hamilton operator one assumes that the electron is orbiting in the (x, y)-plane in a constant magnetic field $\mathbf{K} = \partial_z$ that is directed in the z-direction and derived from the vector potential

$$\mathbf{a} = -\frac{1}{2}(x\partial_y - y\partial_x). \tag{26}$$

Then $\mathbf{K} = \nabla \times \mathbf{a} = \partial_z$ holds. There is also supposed that $\mathbf{E} = 0$, from which $\phi = 0$ follows. The energy of the constant magnetic field provided by the last integral term of (25) is obviously infinity. This integral is infinity also for Coulomb electric fields $\mathbf{E}$ even if $\mathbf{K} = 0$. These were the first infinities appearing in the early days of Quantum Field Theory which were followed by many others in the history of Quantum Theory. Without this confusing term the Zeeman Hamiltonians, function and operator, are of the form (cf. [Bo], formula (15.26.49)):

$$H_z = \frac{1}{2M}|\mathbf{p} + \frac{eB}{c}J_z(x, y)|^2 + V, \tag{27}$$

$$H_Z = \rho(H_z) = -\frac{\hbar^2}{2M}\Delta - \frac{\hbar eB}{2Mc}\mathbf{i}D_z \bullet + \frac{e^2B^2}{8Mc^2}(x^2 + y^2) + V. \tag{28}$$

So far the constant magnetic field was externally specified and not attributed to the charged particle itself. If one supposes that the particle has "built in" angular momentum then the influence of the magnetic dipole moment

$$\mathcal{S} = -\frac{e}{2Mc}\mathcal{L} \tag{29}$$

associated with the angular momentum

$$\mathcal{L} = h(J_x = yp_z - zp_y, J_y = xp_z - zp_x, J_z = xp_y - yp_x) \tag{30}$$

will be felt on subjecting the particle to a magnetic field. The first experiment to observe the magnetic moment of the electron directly is due to Stern and Gerlach (1922), justifying the hypothesis of a "built in" orbital angular momenta in these particles. Note that the angular momentum operator in the Zeeman operator commutes with the rest part, $\mathbf{O}$, of H_Z. Thus the spectrum appears on common eigenfunctions, meaning that the angular momentum operator splits the spectral lines of operator $\mathbf{O}$. This so called Zeeman effect was observed by the Stern-Gerlach experiment, which proved that the angular momentum is quantized.

Still in this chapter the H_Z for a free particle, $V = 0$, is established as the Laplacian on a Riemannian manifold. An interesting feature of this

interpretation is that there will be a well defined finite constant term corresponding to the controversial electromagnetic self energy in the above formulas. In the Point-Spread Theory developed later in this paper this term, H_f, corresponds to the field energy of the "little magnet inside of the point-spread". This combined particle-field Hamiltonian will be denoted by $H_{Zf} = H_Z + H_f$.

2D-Zeeman operators and extended Fock representations. The above Hamiltonians are considered in the literature mostly on $\mathbf{R}^3$, meaning that operator Δ is the Laplacian on the 3-space. However, also the restriction of this 3-dimensional operator to the (x, y)-plane is well known in the literature. Note that (27) becomes a 2D-operator by considering the 2D-Laplacian $\Delta_{(x,y)}$. Then the 3- and 2-dimensional versions differ from each other just by the operator $-(\hbar^2/2\mu)\partial_z^2$, thus the spectral computations with respect to these two cases can be easily compared. The 2D-operators have intensely been investigated, since its first appearance in [AC], in connection with the *Aharonov-Bohm (AB) effect* [AB]. This AB-phenomena got a lot of attention in the near past. A brief account on this problem can be found in [Sz5].

The 2D-Zeeman operators can directly be established by complex Heisenberg group representations. Complex Heisenberg groups have a natural representation on the Hilbert space $\mathcal{H} = L^2_{\mathbf{C}\eta}$ of complex valued functions depending on both of the canonical coordinates (q_i, p_i), where the density η, defining the inner product $\langle f, g\rangle = \int f\overline{g}\eta dX$, is $\eta = e^{-|X|^2}$ or, in general, $\eta_\lambda = e^{-\lambda|X|^2}$, where $\lambda > 0$ is a real constant. Even more general density is introduced later. Comparing with the real case, one should point out numerous other differences.

In the complex case the canonical coordinates are established such that $\partial_{p_i} = J(\partial_{q_i})$ holds. Thus, one can define the holomorphic coordinates $z_i = q_i + \mathbf{i}p_i$. For recreating the whole Hilbert space, $\mathcal{H} = L^2_{\mathbf{C}\eta}$, one should introduce also the antiholomorphic coordinates $\overline{z}_i$. Then the density is, in the simplest case, of the form $\eta = e^{-\sum z_i\overline{z}_i}$, whose most general form is $\eta_{\lambda_i} = e^{-\sum \lambda_i z_i\overline{z}_i}$. The Hilbert space is spanned by the polynomials written in terms of both holomorphic and antiholomorphic coordinates. This Hilbert space is isomorphic to the standard Hilbert space $L^2_{\mathbf{C}}$ having the standard Euclidean density $\eta = 1$ by the map

$$L^2_{\mathbf{C}\eta} \to L^2_{\mathbf{C}} \quad , \quad \psi \to \psi e^{-\frac{1}{2}\sum \lambda_i z_i\overline{z}_i}. \tag{31}$$

Differentiations ∂_{z_i} , $\partial_{\overline{z}_i}$ are defined by means of partial differentiations

∂_{q_i} and ∂_{p_i} by

$$\partial_{z_i} = \frac{1}{2}\partial_{q_i - \mathbf{i}p_i} \quad , \quad \partial_{\overline{z}_i} = \frac{1}{2}\partial_{q_i + \mathbf{i}p_i}. \tag{32}$$

Then the complex Poisson bracket is introduced by

$$\{f, g\}_{\mathbf{C}} = \sum \partial_{z_i}(f)\partial_{\overline{z}_i}(g) - \partial_{\overline{z}_i}(f)\partial_{z_i}(g). \tag{33}$$

The complex Heisenberg algebra is defined by restricting this bracket onto the linear space spanned by the functions $\{z_i, \overline{z}_i, 1\}$ (now the set of holomorphic and antiholomorphic coordinates are extended by the constants $c \in \mathbf{C}$). The real Heisenberg algebra is hidden inside of this complex algebra which can be uncovered by the formulas

$$q_i = \frac{1}{2}(z_i + \overline{z}_i) \quad , \quad p_i = -\frac{1}{2}\mathbf{i}(z_i - \overline{z}_i). \tag{34}$$

The representation of this complex Heisenberg algebra is introduced by

$$\rho_{\mathbf{c}}(z_i)(\psi) = (-\partial_{\overline{z}_i} + \lambda_i z_i \cdot)\psi \quad , \quad \rho_{\mathbf{c}}(\overline{z}_i)(\psi) = \partial_{z_i}\psi. \tag{35}$$

This representation also satisfies the corresponding Heisenberg relations (22). It should be emphasized, however, that this representation is not unitary on the whole complex algebra. It becomes unitary by restricting it onto the real sub-algebra. This statement follows from formulas

$$(-\partial_{\overline{z}_i} + \lambda_i z_i \cdot)^* = \partial_{z_i} \quad , \quad (\partial_{z_i})^* = -\partial_{\overline{z}_i} + \lambda_i z_i \cdot , \tag{36}$$

since one has:

$$\rho_{\mathbf{c}}(q_i) = \frac{1}{2}(\partial^*_{z_i} + \partial_{z_i}) \quad , \quad \rho_{\mathbf{c}}(p_i) = -\frac{1}{2}\mathbf{i}(\partial^*_{z_i} - \partial_{z_i}). \tag{37}$$

Thus $e^{\mathbf{i}\rho_{\mathbf{C}}}$ defines, indeed, a unitary representation of the real subalgebra.

The most important difference between the real and complex representations is that the complex representation $\rho_{\mathbf{C}}$ is not an irreducible representation on the whole Hilbert space $\mathcal{H} = L^2_{\mathbf{C}\eta}$. Indeed, the holomorphic subspace $\mathcal{H}^{(0)}$ spanned by the holomorphic polynomials $z_1^{a_1} \dots z_k^{a_k}$ is obviously invariant and irreducible under the actions of operators determined by the representation. In the literature this irreducible representation is called complex (Fock) representation and, up-to the knowledge of this author, no thorough investigation of the whole reducible representation has been implemented so far. For the sake of clarity, we call it *extended Fock representation*. The 2D-Zeeman operator H_Z, described in (27), can be established from the Hamilton function H_z by the correspondence principle.

Zones established by Gram-Schmidt orthogonalization. Next Hilbert space $\mathcal{H}$ is decomposed into the direct sum of pairwise orthogonal semi-irreducible invariant subspaces. The main tool used in this decomposition is the standard Gram-Schmidt orthogonalization. It turns out, later, that this decomposition is exactly the one established by explicit spectrum computation such that the eigenfunctions are sorted into zones according to their magnetic quantum states. In order to make clear distinction between these two constructions, the latter's are called *spectrally defined Zeeman zones*, while the ones introduced in this chapter are the so called *Heisenberg-Zeeman zones*.

The *Gram-Schmidt process* is applied to the series $G^{(a)}$, $a = 0, 1, \ldots$, of subspaces, where $G^{(a)}$ is spanned by the subspaces $\overline{z}_1^{a_1} \ldots \overline{z}_k^{a_{k/2}} \mathcal{H}^{(0)}$ satisfying $a_1 + \ldots + a_{k/2} = a$. Clearly, $\mathcal{H}^{(0)} = G^{(0)}$ holds. The next subspace, $\mathcal{H}^{(1)}$, is defined as orthogonal complement of $\mathcal{H}^{(0)}$ in $G^{(0)} \oplus G^{(1)}$. The higher order subspaces are defined inductively. Thus, $\mathcal{H}^{(a)}$ is defined as orthogonal complement of $G^{(0)} \oplus \ldots \oplus G^{(a-1)}$ in $G^{(0)} \oplus \ldots \oplus G^{(a)}$.

Since the subspaces $G^{(0)} \oplus \ldots \oplus G^{(a)}$ are obviously invariant under the actions of operators defined in (35), by induction and by the above unitary property one gets that the *subspaces $\mathcal{H}^{(a)}$ are invariant with respect to the complex Heisenberg algebra representation*. The irreducibility is well known on the holomorphic zone. If $k = dim(\mathbf{v}) = 2$, the irreducibility can be established also on the zones of higher order by the same simple proof. Whereas, in case of $k > 0$, the higher order zonal functions defined by a term $\overline{z}_1^{a_1} \ldots \overline{z}_k^{a_{k/2}}$ defined with fixed exponents $a_1, \ldots, a_k$ form an irreducible invariant subspace denoted by $\mathcal{H}^{(a_1 \ldots a_{k/2})} \subset \mathcal{H}^{(a)}$.

The subspaces $\mathcal{H}^{(a)}$ are called *gross zones* on which the formulas appear in a much simpler form than on the irreducible zones. Each gross zone decomposes into $\binom{a+(k/2)-1}{a} = \binom{a+(k/2)-1}{(k/2)-1}$ number of irreducible zones. The gross zones are independent of the complex coordinate system $(z_1, \ldots, z_{k/2})$, while the irreducible zones do depend on them.

Mathematical modeling; Zeeman manifolds. One of the most surprising features of the Zeeman operator H_Z is that it can be pinned down as the Laplacian on a Riemannian manifold. As far as the author knows, this interpretation has not been recognized in the literature so far. We use this Riemannian manifold as the fundamental mathematical model describing the space-time on quantum level. Although the metric is positive definite, this model fulfills the relativistic criteria [Sz5].

By the first version of these manifolds single Zeeman-, or, Pauli-particles are modeled. This fundamental Zeeman manifold is a Riemannian cir-

cle bundle over $\mathbf{R}^2$, defined by factorizing the center of the 3-dimensional Heisenberg group endowed with a left-invariant metric.

The Lie algebra $\mathbf{n} = \mathbf{R}^2 \times \mathbf{R} = \mathbf{R}^3$ (where $\mathbf{R}$ is the center) of the 3D-Heisenberg group can be described in terms of the natural complex structure J, acting on $\mathbf{R}^2$, and the natural inner product $\langle , \rangle$, defined on $\mathbf{n} = \mathbf{R}^3$, by the formula $\langle [X, Y], Z\rangle = \langle tJ(X), Y\rangle$, where the 3-vectors X, Y and Z are in $\mathbf{R}^2$ and $\mathbf{R}$ respectively, furthermore, t is the coordinate of Z in $\mathbf{R}$. The map $Z \to tJ = J_Z$ associates skew endomorphisms acting on $\mathbf{R}^2$ to the elements, Z, of the center. They satisfy the relation $J_Z^2 = -|Z|^2 id$. Thus the metric Lie algebra is completely determined by the system

$$\{\mathbf{n} = \mathbf{v} \oplus \mathbf{z}, \langle , \rangle, J_Z\}, \tag{38}$$

where $\mathbf{v} = \mathbf{R}^2$ and $\mathbf{z}$ are called X- and Z-space respectively. With higher dimensional X- and Z-spaces this system defines the Heisenberg type Lie algebras introduced by Kaplan [Ka]. If the Clifford condition $J_Z^2 = -|Z|^2 id$ is dropped for the skew endomorphisms, the above system defines a most general 2-step nilpotent Lie algebra. The considerations will be extended to these general cases, however, the discussion proceeds with the fundamental 3-dimensional case.

Note that there are two options, J or $J' = -J$, for choosing a complex structure on $\mathbf{R}^2$. The two Lie algebras, $\mathbf{n}$ and $\mathbf{n}'$ are isometrically isomorphic by the map $(X, Z) \to (X, Z' = -Z)$.

The Lie group defined by this Lie algebra is denoted by N, furthermore, g is the left-invariant extension of the inner product $\langle , \rangle$ defined on the tangent space $T_{(0,0)}(N) = \mathbf{n}$ at the origin. Then the exponential map is a one-to-one map whose inverse identifies the group N with its Lie algebra $\mathbf{n}$. Thus also the group lives on the same linear space (X, Z) and the group multiplication is given by:

$$(X, Z)(X^*, Z^*) = (X + X^*, Z + Z^* + \frac{1}{2}[X, X^*]). \tag{39}$$

On the linear coordinate systems $\{x^1, x^2, t\}$, defined by the natural basis $\{E_1, E_2, e_t\}$, the left-invariant extensions of the vectors $E_i; e_t$ are of the form

$$\mathbf{X}_i = \partial_i + \frac{1}{2}\langle [X, E_i], e_t\rangle \partial_t = \partial_i + \frac{1}{2}\langle J(X), E_i\rangle \partial_t \quad ; \quad \mathbf{T} = \partial_t, \tag{40}$$

where $\partial_i = \partial/\partial x^i$, $\partial_t = \partial/\partial t$. Then for the Laplacian, Δ, acting on functions we have:

$$\Delta = \Delta_X + (1 + \frac{1}{4}|X|^2)\partial_{tt}^2 + \partial_t D\bullet, \tag{41}$$

where Δ_X is the Euclidean Laplacian on the X-space and $D\bullet$ means differentiation (directional derivative) with respect to the vector field

$$D : X \to J(X) \tag{42}$$

tangent to the X-space [Sz2, Sz3].

The above Laplacian is not the desired Zeeman operator yet. This surprising interpretation can be established on center-periodic Heisenberg groups defined by an L-periodic lattice $\Gamma_Z = \{Z_{\gamma L} = \gamma L | \gamma \in \mathbf{Z}\}$ on the center. Since the Γ_Z is a discrete subgroup of isometries, one can consider the factor manifold $\Gamma_Z \backslash N$ with the factor metric. The factor manifold is a principal circle bundle over the base space $\mathbf{v}$ such that the circles $C_X = \pi^{-1}(X)$ over the points $X \in \mathbf{v}$ are of constant length L. Then the projection $\pi : \Gamma_Z \backslash N \to \mathbf{v}$ projects the inner product from the horizontal subspace (defined by the orthogonal complement to the circles) to the Euclidean inner product $\langle , \rangle$ on the X-space.

By using the Fourier-Weierstrass decomposition

$$L^2(\Gamma \backslash N) = \oplus FW^{(\gamma)}, \tag{43}$$

where $FW^{(\gamma)}$ consists of functions of the form

$$\phi^{(\gamma)}(X, Z) = \varphi(X) e^{\mathbf{i}\gamma 2\pi t/L}, \tag{44}$$

the Laplacian can be established in the following particular form.

By (41), the function spaces FW^{γ} are invariant under the action of the Laplacian. More precisely we have:

$$\Delta\phi^{(\gamma)} = (\Box_{(\lambda)}\varphi) e^{\mathbf{i}\gamma 2\pi t/L}, \qquad \text{where} \tag{45}$$

$$\Box_{(\lambda)} = \Delta_X + 2\mathbf{i} D_\lambda \bullet - 4\lambda^2 (1 + \frac{1}{4}|X|^2) \quad , \quad \lambda = \frac{\pi\gamma}{L}, \tag{46}$$

and $D_\lambda \bullet = \lambda D \bullet$ means directional derivative along the the vector field $X \to \lambda J(X) = J_\lambda(X)$. If $\lambda < 0$, the J and λ are exchanged for $-J$ and $-\lambda$ respectively. Thus one can assume that $\lambda > 0$.

Apart from the constant term $-4\lambda^2$, operator $\Box_{(\lambda)}$ is nothing but the Zeeman operator H_Z described in (27). The surplus constant term will be identified later with the field-energy of the constant magnetic field in the charge spread, thus the above operator is, actually, H_{Zf} described earlier. The precise description of identification of the Zeeman operator (27) with the Laplacian $\Box_\lambda$ acting on the invariant subspace FW^{γ} is as follows. The macroscopic Zeeman operator is defined by $\hbar = \mu = 1$. Then $H_{Zf} =$

$-(1/2)\Box_{(\lambda)}$, where $\lambda = -eB/2c$. Note that particles with negative charge correspond to the cases $\gamma > 0$, i. e., they are attached to J, while particles with positive charges are attached to $-J$.

On the quantum (microscopic) level, the periodicity L and the parameter λ are exchanged for $L_\hbar = \hbar L$ and $\lambda_\hbar = \lambda/\hbar$ respectively. This process means nothing but scaling of the periodicity by $\hbar$. Then we have $H_{Zf} = -(\hbar^2/2\mu)\Box_{(\lambda_\hbar)}$. By scaling also the Euclidean metric on the X-space by $\hbar/\sqrt{\mu}$, one has $H_{Zf} = -(1/2)\Box_{(\lambda_\hbar)}$. In the following we proceed with the macroscopic operator, however, the previous formulas allow an easy transfer from the macroscopic level to the microscopic one.

By a straightforward generalization, described later, these operators can be introduced on higher dimensional Heisenberg groups defined by a complex structure, J, on an even dimensional Euclidean space $\mathbf{R}^k$. Let $(z_1, \ldots, z_{k/2})$, where $z_i = q_i + \mathbf{i}p_i$ and $\partial_{p_i} = J(\partial_{q_i})$, be a complex coordinate system regarding J. Then this system identifies $\mathbf{R}^k$ with $\mathbf{C}^{k/2}$. The circle bundle, defined by factorizing the center, $\mathbf{R}$, determines quantum operators depending just on one parameter λ. One can easily introduce operators depending on different parameters $\lambda_i > 0$ defined for each complex coordinate plane z_i. (Such operators are constructed in the next section.) These operators correspond to the Hamiltonians of systems where $k/2$ number of charged particles are circulating in a constant magnetic field. When there is only one λ involved, the particles are considered to be identical up-to the sign of the charge.

Zeeman manifolds with higher dimensional centers. The mathematical model for interpreting the Zeeman operator as the Laplacian on a Riemannian manifold has been, so-far, a Riemannian circle bundle, defined by factorizing the centers on Heisenberg groups endowed with left invariant metrics. This idea works out also on metric two-step nilpotent Lie groups whose center, $\mathbf{z}$, is factorized by a lattice Γ_Z. This center is considered as an abstract higher dimensional space such that an element $Z \in \mathbf{z}$ is identified with the endomorphism $J_Z : \mathbf{v} \to \mathbf{v}$ and its natural inner product is defined by $\langle Z_1, Z_2\rangle = -Tr(J_{Z_1} \circ J_{Z_2})$. Formulas (38)-(41) apply also to these general cases, just the Laplacian (41) appears in a slightly different form. Up-to isomorphism, the Lie algebra of such a group is uniquely determined by a linear space, $J_{\mathbf{z}}$, of skew endomorphisms acting on the Euclidean space $\mathbf{v}$. Two 2-step nilpotent groups are isometrically isomorphic if and only if the corresponding endomorphism spaces are conjugate.

The rather large class of Riemannian torus bundles introduced in this way are called also Zeeman manifold. Below also particular Zeeman mani-

folds are introduced. It is remarkable that for the so called Clifford-Zeeman manifolds even classification can be implemented. This classification can be used for classifying the charged particles investigated in this theory.

The Laplacian on the Riemannian group $(N_{J_{\mathbf{z}}}, g)$, defined by the endomorphism space $J_{\mathbf{z}}$, has the explicit form:

$$\Delta = \Delta_X + \Delta_Z + \frac{1}{4}\sum_{\alpha,\beta=1}^{r}\langle J_\alpha(X), J_\beta(X)\rangle\partial^2_{\alpha\beta} + \sum_{\alpha=1}^{r}\partial_\alpha D_\alpha \bullet, \tag{47}$$

which leaves the function spaces $FW^{(\gamma)}$ spanned by the functions of the form $\Psi^{(\gamma)}(X,Z) = \psi(X)e^{2\pi\mathbf{i}\langle\mathcal{Z}_\gamma,Z\rangle} = \psi(X)e^{2\mathbf{i}\langle Z_\gamma,Z\rangle}$, for all lattice points $\mathcal{Z}_\gamma \in \Gamma_Z$ (resp. $Z_\gamma \in \pi\Gamma_Z$), invariant. Its action on such a function space can be described in the form $\Delta(\Psi^{(\gamma)})(X,Z) = \Box_{(\gamma)}(\psi)(X)e^{2\pi\mathbf{i}\langle\mathcal{Z}_\gamma,Z\rangle}$, where operator $\Box_{(\gamma)}$, acting on $L^2(\mathbf{v})$, is of the form

$$\begin{aligned}\Box_{(\gamma)} &= \Delta_X + 2\pi\mathbf{i}D_{(\gamma)} \bullet -4\pi^2(|\mathcal{Z}_\gamma|^2 + \frac{1}{4}|J_{\mathcal{Z}_\gamma}(X)|^2) \\ &= \Delta_X + 2\mathbf{i}D_{Z_\gamma} \bullet -4(|Z_\gamma|^2 + \frac{1}{4}|J_{Z_\gamma}(X)|^2).\end{aligned} \tag{48}$$

Thus the Zeeman operator appears on the invariant subspaces defined by the Fourier-Weierstrass decomposition. The spectral investigations on these manifolds are reduced to investigate this operator on each Fourier-Weierstrass subspace separately.

The particles represented by these Riemannian torus bundles are called *Zeeman molecules*. A Zeeman molecule consists of $k/2$ number of charged particles. The higher dimensional center represents the variety of the individual constant magnetic fields defined for the particles in the molecule. It is explained in [Sz5] that the constant magnetic field fixes an inertia system which defines the self-time for a particle. Thus this general model matches Dirac's famous multi-time model, introduced for implementing relativistic criteria on the quantum level. Lattice Γ in the Z-space relates our model to the crystal models of quantum theory.

There are special Z-molecules, defined by particular endomorphism spaces, which are particularly interesting. The *Heisenberg-type* or *Cliffordian endomorphism spaces* are attached to Clifford modules (representations of Clifford algebras). They are characterized by the property $J_Z^2 = -|Z|^2 id$, for all $Z \in \mathbf{z}$, [Ka]. The corresponding molecules are called *Clifford-Zeeman molecules*. The well known *classification* of Clifford modules provides classification also for the Clifford endomorphism spaces and molecules. A brief account on this classification theorem is as follows.

If $r = dim(J_{\mathbf{z}}) \neq 3(mod4)$, then there exist (up to equivalence) exactly one irreducible H-type endomorphism space acting on a $\mathbf{R}^{n_r}$, where the dimension n_r, depending on r, is described below. This endomorphism space is denoted by $J_r^{(1)}$. If $r = 3(mod4)$, then there exist (up to equivalence) exactly two non-equivalent irreducible H-type endomorphism spaces acting on $\mathbf{R}^{n_r}$ which are denoted by $J_r^{(1,0)}$ and $J_r^{(0,1)}$ respectively. They are connected by the relation $J_r^{(1,0)} \simeq -J_r^{(0,1)}$.

The values n_r corresponding to $r = 8p, 8p+1, \ldots, 8p+7$ are

$$n_r = 2^{4p}\,,\, 2^{4p+1}\,,\, 2^{4p+2}\,,\, 2^{4p+2}\,,\, 2^{4p+3}\,,\, 2^{4p+3}\,,\, 2^{4p+3}\,,\, 2^{4p+3}. \tag{49}$$

The reducible Clifford endomorphism spaces can be built up by these irreducible ones. They are denoted by $J_r^{(a)}$ resp. $J_r^{(a,b)}$. The corresponding Lie algebras are denoted by $h_r^{(a)}$ resp. $h_r^{(a,b)}$. In the latter case the X-space is defined by the $(a+b)$-times product $\mathbf{R}^{n_r} \times \ldots \times \mathbf{R}^{n_r}$ such that on the last b component the action of a J_Z is defined by $J_Z^{(0,1)}$ and on the first a components this action is defined by $J_Z^{(1,0)}$. In the first case this process should be applied only on the corresponding a-times product.

In a Clifford endomorphism space each endomorphism anticommutes with all perpendicular endomorphisms. In other words, all endomorphisms are anticommutators. A more general concept can be introduced by the *anticommutative endomorphism spaces* where all endomorphisms are anticommutators. They can be built up, in a non-trivial way, by Clifford endomorphism spaces. Roughly speaking, a CZ-molecule is the compound of irreducible molecules of the same type while an *anticommutative Z-molecule* is an indecomposable compound of CZ-molecules of different types in general.

Originally, the metric groups (N_J, g) were used, in many different ways, for constructing isospectral Riemannian metrics with different local geometries. The author's results regarding such constructions are published in [Sz1, Sz2, Sz3, Sz4] which contain also detailed history about this topic. These examples include isospectral pairs of metrics on ball×torus-, sphere×torus-, ball-, and sphere-type manifolds. Among these examples the most striking are those where one of the metrics in the isospectral pair is homogeneous while the other is not even locally homogeneous. Such examples have been constructed so-far on sphere-, sphere×sphere, and sphere×torus-type manifolds. These isospectrality constructions are implemented such that on some of the irreducible subspaces $\mathbf{R}^{n_r}$ the endomorphism spaces $J_r^{(1,0)}$ (resp.

$J_r^{(0,1)}$) are switched to $J_r^{(0,1)}$ (resp. $J_r^{(1,0)}$). It turns out that the Riemannian space, resulted by this switching, has a completely different local geometry, yet, the considered domains in the original and the new Riemann spaces are isospectral. Endomorphism spaces $J_r^{(1,0)}$ and $J_r^{(0,1)}$ are considered to be representing irreducible CZ-particles having opposite charges. They are called also antiparticles. Thus the isospectrality theorem can be physically interpreted as follows:

By exchanging some of the irreducible CZ-particles in a CZ-molecule with their antiparticles the spectra of the considered domains remain the same, however, the local geometry is drastically changed in general.

Most of these isospectrality statements are established by constructing intertwining operators, while some are proved by explicit computations of the spectrum. These computations are different from the one developed for the Zeeman zones.

3 Zonal projections, zonal point-spreads

Establishing the projection kernels. In this section the operators $\mathbf{P}^{(a)}$ projecting onto the zones $\mathcal{H}^{(a)}$ will be explicitly described. If $\{\varphi_i^{(a)}\}_{i=1}^{\infty}$ is an orthonormal basis in $\mathcal{H}^{(a)}$, then the corresponding projection can be formally defined as convolution with the kernel

$$P^{(a)}(z,w) = \sum_i \varphi_i^{(a)}(z)\overline{\varphi_i^{(a)}(w)}. \tag{50}$$

Interestingly enough, these self-adjoint operators are integral operators having a smooth Hermitean integral kernel $P^{(a)}(z_j, w_j)$. The projection kernel regarding the holomorphic zone is the well known Bergman kernel. However, up-to the knowledge of this author, the projections onto the other zones have never been considered in the literature so far. First, we describe these kernels for the 2D-Zeeman operator defined on the complex plane $\mathbf{C}$ (i. e., $k = 2$). Note that in this case only a single parameter λ is involved to the computations.

On the zone $\mathcal{H}^{(0)}$ the holomorphic polynomials z^i form an orthogonal basis, thus the holomorphic polynomials

$$\varphi_i^{(0)}(z) = \mathbf{z}^i = (1/\int (z\overline{z})^i d\eta)^{\frac{1}{2}} z^i \tag{51}$$

form an orthonormal basis in $\mathcal{H}^{(0)}$. The sought integral operator (convolu-

tion operator) on this subspace is

$$P^{(0)}(z,w) *_w f(w) = \int P^{(0)}(z,w) f(w) d\eta(w) = \tag{52}$$
$$\int \frac{1}{\pi} e^{z\overline{w}} f(w) e^{-w\overline{w}} dw = \int \frac{\lambda}{\pi} e^{\lambda z\overline{w}} f(w) e^{-\lambda w\overline{w}} dw,$$
$$\mathbf{P}^{(0)}_{\lambda}(f)(Z) = \int \frac{\lambda}{\pi} e^{\lambda(\langle Z,W\rangle + \mathbf{i}\langle Z,J(W)\rangle)} f(W) e^{-\lambda|W|^2} dW, \tag{53}$$

where Z, W denotes the the corresponding complex numbers as vectors and J is the complex structure on this vector space. (The latter formula is used for an easy generalization to the higher dimensional cases. If the real dimension is k and there is a single λ involved, only the constant should be changed to $(\lambda/\pi)^{k/2}$.) Formula (52) can be established by showing that the image space of this Hermitean integral operator is $\mathcal{H}^{(0)}$ and also $P^{(0)}(z,w) *_w w^i = w^i$ holds.

Now we pass to the next zone, $\mathcal{H}^{(1)}$. Also this case is considered on the complex plane $\mathbf{C}$ first. Now the functions

$$\phi^{(1)}_i(z) = \overline{z} z^i - \mathbf{P}^{(0)}(\overline{z} z^i) = \overline{z} z^i - (1/\lambda) i z^{i-1} \tag{54}$$

form an orthogonal basis on the zone and the normalization provides an orthonormal basis. Also in this case the projection $P^{(1)}$ is an integral operator whose kernel can be constructed as follows.

First note that this projection can be described by the generalized kernel

$$P^{(1)}_{zw} = (\overline{z} P^{(0)}_z) *_w \partial_{\overline{w}} - P^{(0)}_z *_v (\overline{v} P^{(0)}_v) *_w \partial_{\overline{w}} \tag{55}$$

since the range of this Hermitean operator is $\mathcal{H}^{(1)}$ which acts on this subspace as the identity. Also this operator, which still contains differentiations, is an integral operator with a smooth Hermitian kernel. This statement follows by the following computations:

$$P^{(1)}_{zw} = (\overline{z}(w-z) P^{(0)}_z) *_w -(\overline{w} P^{(0)}_z) *_w \partial_{\overline{w}} = \tag{56}$$
$$((\overline{z}(w-z) + 1 + \overline{w}(z-w)) P^{(0)}_z) *_w = ((1 - |z-w|^2) P^{(0)}_z) *_w ,$$
$$\mathbf{P}^{(1)}(f)_Z = \int \frac{1}{\pi} (1 - |Z-W|^2) e^{(\langle Z,W\rangle + \mathbf{i}\langle Z,J(W)\rangle)} f(W) e^{-|W|^2} dW. \tag{57}$$

The derivative $\partial_{\overline{w}}$, acting on functions f first, is handled by integration by parts such that it is substituted by its dual's action on the function $(\overline{z} P^{(0)}_z) - P^{(0)}_z *_v (\overline{v} P^{(0)}_v)$.

Passing to the second zone, $\mathcal{H}^{(2)}$, the same computational technique, applied in a more complicated situation, yields the following formulas:

$$\begin{aligned} 2!\mathbf{P}_z^{(2)}(f) &= ((\overline{z}^2 P_z^{(0)}) *_w \partial_{\overline{w}}^2 \\ -(P_z^{(0)} + P_z^{(1)}) *_v (\overline{v}^2 P_v^{(0)}) *_w \partial_{\overline{w}}^2)(f) &= \{\overline{z}^2(w-z)^2 P_z^{(0)} *_w \\ -\overline{v}^2(w-v)^2(2-(z-v)(\overline{z}-\overline{v}))P_z *_v P_v *_w\}(f) \\ = \overline{z}^2(w-z)^2 P_z^{(0)} *_w -\partial_v^2\{(w-v)^2(1-(z-v)(\overline{z}-\overline{v})) \\ -2(z-v)(w-v)\}P_z *_v P_v *_w &= \{\overline{z}^2(w-z)^2 + (w-z)^2\overline{w}^2 \\ -2\overline{w}\overline{z}(w-z)^2 - 4(z-w)(\overline{z}-\overline{w}) + 2\}P_z^{(0)} *_w (f) \\ = \int \frac{1}{\pi}(2-4|Z-W|^2+|Z-W|^4)e^{Z\cdot\overline{W}} f(W) e^{-|W|^2} dW. \end{aligned} \tag{58}$$

Term $Z \cdot \overline{W}$ often appears later in the form $Z \cdot \overline{W} = \langle Z, W\rangle + \mathbf{i}\langle Z, J(W)\rangle$.

Regarding the polynomials arising in these formulas, note that $L_0^{(0)}(t) = 1, L_1^{(0)}(t) = 1-t, L_2^{(0)}(t) = (2-4t+t^2)/2!$ are the first three Laguerre polynomials parametrized by $\alpha = 0$ and one gets the above ones by the substitution $t = |Z-W|^2$. These polynomials depend, in general, on a parameter α [Sze], such that the n^{th}-order polynomials $L_n^{(\alpha)}$ are defined by the eigenfunctions of the operator

$$\Lambda_\alpha(u)(t) = tu'' + (\alpha + 1 - t)u'. \tag{59}$$

The corresponding eigenvalue is $-n$. The above polynomials belong to parameter $\alpha = 0$. On the k-dimensional case this parameter will be $\alpha = (k/2) - 1$.

An analogous definition for these polynomials uses the following Rodrigues formula:

$$e^{-t}t^\alpha L_a^{(\alpha)}(t) = \frac{1}{a!}\partial_t^a(e^{-t}t^{a+\alpha}). \tag{60}$$

The well known explicit form of these polynomials is

$$L_a^{(\alpha)}(t) = \sum_{b=0}^{a} \binom{a+\alpha}{a-b} \frac{(-t)^b}{b!}. \tag{61}$$

The followings recurrence formulas will be often used in establishing the general projection kernels.

$$(a+1)L_{a+1}^{(\alpha)}(t) = (\alpha + 2a + 1 - t)L_a^{(\alpha)}(t) - aL_{a-1}^{(\alpha)}(t)\,, \tag{62}$$

$$\sum_{b=0}^{a} L_b^{(\alpha)} = L_a^{(\alpha+1)}(t)\,,\ \partial_t L_a^{(\alpha)} = -L_{a-1}^{(\alpha+1)}\,, \tag{63}$$

$$tL_a^{(\alpha)\prime}(t) = aL_a^{(\alpha)}(t) - (a+\alpha)L_{a-1}^{(\alpha)}(t)\,. \tag{64}$$

The pre-formula (55) for defining the projections can be recurrently introduced for the higher order cases. Indeed, if $P^{(0)}, \ldots, P^{(a-1)}$ are already defined, the pre-formula for $P^{(a)}$ is:

$$\mathbf{P}^{(a)}(f)_z = \frac{1}{a!}((\overline{z}^a P^{(0)}) *_w -(P_z^{(0)} + \ldots + P_z^{(a-1)}) *_v (\overline{v}^a P^{(0)}) *_w)\partial_{\overline{w}}^a(f_w)(65)$$

Strictly speaking, these formulas concern the case $k = 2\,,\, \lambda = 1$. If for a general k; but still assuming $\lambda = 1$; the corresponding projections with respect to a complex coordinate z_i are denoted by $\mathbf{P}^{(a_i)}$, then the sought pre-formula is:

$$\mathbf{P}^{(a)} = \sum_{a_1+\ldots+a_{k/2}=a} \mathbf{P}^{(a_1)} \ldots \mathbf{P}^{(a_{k/2})}. \qquad (66)$$

The long computation in (58) shows that a direct establishing of the Laguerre polynomial form of the projection operators would be an arduous task. This is why mathematical induction has been chosen below for concluding the proof of the corresponding theorem.

Theorem 3.1 (Zonal Theorem) *Representation (35) is a reducible unitary representation of the complex Heisenberg algebra. The function space $L^2_{\mathbf{C}\eta}$ is an orthogonal direct sum of invariant subspaces $\mathcal{H}^{(a)}$, called gross Zeeman zones. In case $k = 2$, this representation is irreducible on each zone. It is always irreducible also on the holomorphic zones, however, it is reducible if $k > 2$ and $a > 0$. (The irreducible zones are considered in the end of this theorem.)*

One can construct the gross zone decomposition by the Gram-Schmidt orthogonalization of subspaces $G^{(a)} = \overline{z}_1^{a_1} \ldots \overline{z}_k^{a_k}\mathcal{H}^{(0)}$, where $a_1 + \ldots + a_k = a$. The first zone $\mathcal{H}^{(0)} = G^0$ is the holomorphic zone spanned by the holomorphic polynomials $z_1^{a_1} \ldots z_k^{a_k}$.

The projections onto the gross zones are integral operators having analytic integral kernels. In case of $k = 2\,,\, \lambda = 1$ these integral operators are

$$\mathbf{P}^{(a)}(f)_Z = \int \frac{1}{\pi} L_a^{(0)}(|Z - W|^2) e^{\langle Z,W\rangle + \mathbf{i}\langle Z,J(W)\rangle} f(W) e^{-|W|^2} dW, \qquad (67)$$

where $L_a^{(0)}(t)$ is the corresponding Laguerre polynomial. In the k-dimensional cases, where there is only a single parameter $\lambda > 0$ involved, the corresponding integral operators are

$$\mathbf{P}_\lambda^{(a)}(f)_Z = \int \frac{\lambda^{k/2}}{\pi^{k/2}} L_a^{((k/2)-1)}(\lambda|Z - W|^2) e^{\lambda Z\cdot\overline{W}} f(W) e^{-\lambda|W|^2} dW, \qquad (68)$$

where $Z \cdot \overline{W} = \langle Z, W\rangle + \mathbf{i}\langle Z, J(W)\rangle$. *If more* λ_i*'s are involved which are defined for* k_i*-dimensional subspaces such that* $k = \sum_{i=1}^{p} k_i$ *holds then the projection is:*

$$\mathbf{P}^{(a)}_{\lambda_1 \dots \lambda_p}(f)_Z = \quad (69)$$

$$\int \frac{\prod \lambda_i^{k_i/2}}{\pi^{k/2}} L_a^{((k/2)-1)}(\sum \lambda_i |Z_i - W_i|^2) e^{\sum \lambda_i Z_i \cdot \overline{W}_i} f(W) e^{-\sum \lambda_i |W_i|^2} dW.$$

Each gross zone decomposes into $\binom{a+(k/2)-1}{a} = \binom{a+(k/2)-1}{(k/2)-1}$ *number of irreducible zones. The gross zones are independent of the complex coordinate system* $(z_1, \dots, z_{k/2})$*, while the irreducible zones do depend on them. The irreducible zones, denoted by* $\mathcal{H}^{(a_1 \dots a_{k/2})} \subset \mathcal{H}^{(a)}$*, are spanned by the higher order zonal functions defined by the antiholomorphic terms* $\overline{z}_1^{a_1} \dots \overline{z}_k^{a_{k/2}}$ *for fixed exponents* $a_1, \dots, a_k$*. The projection,* $\mathbf{P}^{(a_1 \dots a_{k/2})}_{\lambda_1 \dots \lambda_p}(f)_Z$*, onto an irreducible zone differs from the previous operator just in the Laguerre polynomial term what should be exchanged for* $\prod_{i=1}^{k/2} L_{a_1}^{(0)}(\lambda_i |Z_i - W_i|^2)$.

Proof. The proof goes through the cases indicated in the theorem step by step. If $k = 2, \lambda = 1$, then (67) is proved for $a \leq 2$ by (51)-(58). Suppose that it is valid for $a \geq 2$. Then for its validity regarding $a + 1$ one should prove that the integral operator defined by the kernel

$$\mathbf{L}^{(0)}_{a+1}(z, w) = \frac{1}{\pi} L^{(0)}_{a+1}(|z - w|^2) e^{z\overline{w}} \quad (70)$$

is nothing but the projection onto $\mathcal{H}^{(a+1)}$.

The range of this Hermitean operator is in $\mathcal{H}^{(\leq a+1)} = G^{(0)} \oplus \dots \oplus G^{(a+1)}$, therefore, it maps the perpendicular complement $\mathcal{H}^{(\leq a+1)\perp}$ to zero. Thus, one should consider this operator only on $\mathcal{H}^{(\leq a+1)}$. Therefore, it is enough to prove that it maps also the functions $\overline{w}^b w^i \in \mathcal{H}^{(\leq a)}$, where $b \leq a$, to zero. By (62), (64), induction, and integration by parts one can carry out the following computations.

$$\int (a+1) \mathbf{L}^{(0)}_{a+1}(z, w)(\overline{w}^b w^i) d\eta \quad (71)$$

$$= \int ((2a + 1 - |z - w|^2) \mathbf{L}^{(0)}_a(z, w) - a \mathbf{L}^{(0)}_{a-1}(z, w))(\overline{w}^b w^i) d\eta$$

$$= \int ((2a+1) \mathbf{L}^{(0)}_a(z, w) - a \mathbf{L}^{(0)}_{a-1}(z, w))(\overline{w}^b w^i) d\eta$$

$$- \int \partial_w (|z - w|^2 \mathbf{L}^{(0)}_a(z, w)(\overline{w}^{b-1} w^i)) d\eta$$

$$= \int (a\mathbf{L}^{(0)}_a(z,w)(\overline{w}^b w^i) - a\overline{z}\mathbf{L}^{(0)}_{a-1}(z,w)(\overline{w}^{b-1}w^i))d\eta$$

$$-\int(-a\mathbf{L}^{(0)}_{a-1}(z,w)\overline{w}^{b-1}\partial_w w^i + |z-w|^2\mathbf{L}^{(0)}_a(z,w)\partial^b_w(w^i))d\eta = \phi(z)\,.$$

Since the formula is established for $a = 0, 1, 2$, we can suppose $a \geq 2$. Then the integral of the very last function in the last line is zero, since it refers to the $\mathbf{P}^{(a)}$-projection of a function from $\mathcal{H}^{\leq 1}$ onto $\mathcal{H}^{(a)}$. Thus the the ϕ decomposes into three integral terms, $\phi = \phi_1 + \phi_2 + \phi_3$, where ϕ_3 is derived from the last line and the other two from the line before the last one.

If $b < a$, then $\phi = 0$, since the integral vanishes by terms, due to the induction. This means that the $P^{(a+1)} = \mathbf{L}^{(0)}_{a+1}(z,w)*$ operates trivially on $\mathcal{H}^{(\leq a-1)}$.

If $a = b$, then ϕ is in $\mathcal{H}^{(\leq a-1)}$. In fact, the ϕ_3 is obviously in that subspace, furthermore, the first two terms corresponding to $\phi_1 + \phi_2$ is a difference of functions from $\mathcal{H}^{(\leq a)}$ such that they have the same leading term, $a\overline{z}^a z^i$. Thus also this function must be in $\mathcal{H}^{(\leq a-1)}$. Therefore, by applying $P^{(a+1)} = \mathbf{L}^{(0)}_{a+1}(v,z)*$ on ϕ, one gets the zero function: $(P^{(a+1)})^2(\overline{w}^b w^i) = 0$. Since the operator considered is Hermitean, also the ϕ must be zero then. This proves the desired identity.

Proof of (67) will be completed by showing that the integral operator with kernel $\mathbf{L}^{(0)}_{a+1}(z,w)$ acts on $G^{(a+1)}$ as the projection $\mathbf{P}^{(a+1)}$. By the induction hypothesis, we have.

$$\int \mathbf{L}^{(0)}_{a+1}(z,w)\overline{w}^{a+1}w^i d\eta = \overline{z}^{a+1}z^i - \sum_{b=0}^{a}\mathbf{P}^{(b)}(\overline{w}^{a+1}w^i) \tag{72}$$

$$= \overline{z}^{a+1}z^i - \int\sum_{b=0}^{a}\mathbf{L}^{(0)}_b(z,w)\overline{w}^{a+1}w^i d\eta =$$

$$\overline{z}^{a+1}z^i - \int \mathbf{L}^{(1)}_a(z,w)\overline{w}^{a+1}w^i d\eta.$$

The last equation is due to (63). By (62), (64), and the first part of this proof we have

$$\int \mathbf{L}^{(0)}_{a+1}(z,w)\overline{w}^{a+1}w^i d\eta = \int \partial_w(\mathbf{L}^{(0)}_{a+1}(z,w)\overline{w}^a w^i)d\eta = \tag{73}$$

$$\int \mathbf{L}^{(1)}_a(z,w)(\overline{z}-\overline{w})\overline{w}^a w^i d\eta = \overline{z}^{a+1}z^i - \int \mathbf{L}^{(1)}_a(z,w)\overline{w}^{a+1}w^i d\eta.$$

This concludes the proof of (67).

Formula (68), concerning the general k-dimensional cases, will be proved, first, for $\lambda = 1$. In light of (65) and (66) the statement immediately follows from

$$L_n^{(\alpha)}(t_1 + \ldots + t_{\alpha+1}) = \sum_{a_1+\ldots+a_{\alpha+1}=n} L_{a_1}^{(0)}(t_1) \ldots L_{a_{\alpha+1}}^{(0)}(t_{\alpha+1}), \tag{74}$$

which can be readily established by the following simpler formula:

$$L_n^{(\alpha+1)}(t_1 + t_2) = \sum_{a=0}^{n} L_a^{(0)}(t_1) L_{n-a}^{(\alpha)}(t_2). \tag{75}$$

The latter formula can be established by (61) or by mathematical induction. Below the latter method is chosen, since this was the natural way in which (67)-(69) were found. In establishing (75) the induction is used twice. First, in proving of the starting step

$$L_n^{(1)}(t_1 + t_2) = \sum L_a^{(0)}(t_1) L_{n-a}^{(0)}(t_2), \tag{76}$$

where the induction concerns n. Then the induction concerns α in proving of (75).

Since $L_1^{(0)}(t_1) + L_1^{(0)}(t_2) = 2 - (t_1 + t_2)$, formula (76) is valid for $n = 1$. Assume that it is valid for *degrees* $\leq (n-1)$. Then, by applying (62) both for $\alpha = 1$ and $\alpha = 0$, one has the following sequence of equations

$$\begin{aligned}
nL_n^{(1)}(t_1 + t_2) = (2n - (t_1 + t_2))L_{n-1}^{(1)}(t_1 + t_2) - nL_{n-2}^{(1)}(t_1 + t_2) = \qquad (77)\\
\sum_{a=0}^{n-1}((2a + 1 - t_1) + (2(n - 1 - a) + 1 - t_2))L_a^{(0)}(t_1)L_{n-1-a}^{(0)}(t_2) -\\
-nL_{n-2}^{(1)}(t_1 + t_2) = \sum_{a=0}^{n-1}\big((a+1)L_{a+1}^{(0)}(t_1)L_{n-1-a}^{(0)}(t_2) + (n-a)L_a^{(0)}(t_1)L_{n-a}^{(0)}(t_2)\\
+aL_{a-1}^{(0)}(t_1)L_{n-1-a}^{(0)}(t_2) + (n-1-a)L_a^{(0)}(t_1)L_{n-a-2}^{(0)}(t_2)\big) - nL_{n-2}^{(1)}(t_1 + t_2) =\\
= n\sum_{b=0}^{n} L_b^{(0)}(t_1)L_{n-b}^{(0)}(t_2) + n\sum_{c=0}^{n-2} L_c^{(0)}(t_1)L_{n-c-2}^{(0)}(t_2) - nL_{n-2}^{(1)}(t_1 + t_2) =\\
= n\sum_{b=0}^{n} L_b^{(0)}(t_1)L_{n-b}^{(0)}(t_2).
\end{aligned}$$

The terms in the first and second summation formulas are equal by the recurrence equation

$$(c+1)L_{c+1}^{(0)}(t) = (2c + 1 - t)L_c^{(0)}(t) - cL_{c-1}^{(0)}(t). \tag{78}$$

To complete the proof of (75) suppose that it is valid for α. Then its validity for $\alpha+1$ follows by using (63) twice:

$$L_n^{(\alpha+1)}(t_1+t_2)=\sum_{a=0}^{n}L_a^{(\alpha)}(t_1+t_2)=\sum_{a=0}^{n}\sum_{p=0}^{a}L_p^{(\alpha-1)}(t_1)L_{a-p}^{(0)}(t_2) \tag{79}$$

$$=\sum_{b=0}^{n}(\sum_{a=b}^{n}L_{a-b}^{(\alpha-1)}(t_1))L_b^{(0)}(t_2)=\sum_{b=0}^{n}L_{n-b}^{(\alpha)}(t_1)L_b^{(0)}(t_2).$$

Formulas (52) and (74) establish the most general case described in (69) by the substitutions $t_i=\lambda_i|Z_i-W_i|^2$. Finally, the statement concerning the projection kernels regarding the irreducible zones can be established by (74). These arguments complete the proof of the theorem.

□

Point-spreads. Since the zonal projection kernels are the restrictions of the global Dirac delta distribution $\delta_Z(W)=\sum\varphi_i(Z)\overline{\varphi}_i(W)$, they are denoted also in the following form

$$\delta^{(a)}_{\lambda_1\ldots\lambda_{k/2}}(Z,W)=P^{(a)}_{\lambda_1\ldots\lambda_{k/2}}(Z,W)= \tag{80}$$

$$\frac{\prod\lambda_i^{k_i/2}}{\pi^{k/2}}L^{(a)}_{(k/2)-1}(\sum\lambda_i|Z_i-W_i|^2)e^{\sum\lambda_i(Z_i\cdot\overline{W}_i-\frac{1}{2}(|Z_i|^2+|W_i|^2)}.$$

The kernels $\delta^{(a_1\ldots a_{k/2})}_{\lambda_1\ldots\lambda_{k/2}}(Z,W)$ on the irreducible zones $\mathcal{H}^{(a_1\ldots a_{k/2})}\subset\mathcal{H}^{(a)}$ differ from these ones just in the Laguerre polynomial term what should be exchanged for $\prod_{i=1}^{k/2}L_{a_1}^{(0)}(\lambda_i|Z_i-W_i|^2)$.

As opposed to the global Dirac-delta distribution kernel, these ones are smooth functions. They represent one of most important fundamental concepts in this article. Also note the simple rule by which the zonal kernels are derived from the one defined for the holomorphic zone. This holomorphic point-spread is just multiplied by the corresponding radial Laguerre polynomials. These zonal kernels can be interpreted such that a point particle concentrated at a point Z appears on the zone as an object which spreads around Z as a wave packet with wave-function described by the above kernel explicitly.

There is a complete matching between the point-spreads and de Broglie's wave packets (cf. [Bo], pages 61), which concept became one of the corner stones of Quantum Theory. De Broglie's theory was finalized by the Schrödinger equation. The mathematical formalism did not follow this development, however, and the Schrödinger theory is built up on such mathematical background which does not exclude the existence of the controversial

point objects. On the contrary, an electron must be considered as a point-object in the Schrödinger theory as well (cf. Weisskopf's argument on this problem in [Schw, Sz5]). An other demonstration for the presence of point particles in classical theory is the duality principle, stating that objects manifest themselves sometime as waves and sometime as point particles. The bridge between the two visualizations is built up in Born's probabilistic theory, where the probability for that that a particle, attached to a wave ξ, can be found on a domain D is measured by $\int_D \xi\overline{\xi}$.

These controversial point-objects, by having infinite self-mass or self-charge attributed to them by the Schrödinger, launched one of the deepest crisis-es in the history of physics. In the zonal theory de Broglie's idea is established on a mathematical level. Although the points are ostracized from this theory, the point-spreads still bear some reminiscence of the point-particles. For instance, they are the most compressed wave-packets and all the other wave-functions in the zone can be expressed as a unique superposition of the point-spreads. If ξ is a zone-function, the above integral measures the probability for that that the center of a point-spread is on the domain D. This interpretation restores, in some extend, the duality principle in the zonal theory.

Function $\delta^{(a)}_{\lambda Z}\overline{\delta}^{(a)}_{\lambda Z}$ is called the density of the spread around Z. By this reason, function $\delta^{(a)}_{\lambda Z}$ is called spread-amplitude. Both the spread-amplitude and spread-density generate well defined measures on the path-space consisting of continuous curves connecting two arbitrary points. Both measures can be constructed by the method applied in constructing the Wiener measure.

The point-spread concept bears some remote reminiscence of Heisenberg's suggestion (1938) for the existence of a fundamental length L, analogously to h, such that field theory was valid only for distances larger than L and so divergent integrals would be cut off at that distance. This idea has never became an effective theory, however. Other distant relatives of the point-spread concept are the smeared operators, i. e. those suitably averaged over small regions of space-time, considered by Bohr and Rosenfeld in quantum field theory. There are also other theories where an electron is considered to be extended. Most of them fail on lacking the explanation for the question: Why does an extended electron not blow up? The zonal theory is checked against this problem in [Sz5], section (F) "Linking to the blackbody radiation; Solid zonal particles".

4 Spectral definition of zones

Technicalities. In this chapter the zones are established by the explicit spectra of the Zeeman-Laplacians defined on center-periodic 2-step nilpotent manifolds $\Gamma\backslash N$. On these manifolds the Laplacian leaves the subspaces defined by the Fourier-Weierstrass decomposition

$$L^2(\Gamma\backslash N) = \oplus FW^{(\gamma)} \tag{81}$$

invariant. The $FW^{(\gamma)}$ is defined for any fixed lattice point $\mathcal{Z}_\gamma \in \Gamma_Z$ (resp. $Z_\gamma \in \pi\Gamma_Z$) such that it consists the functions of the form

$$\phi^{(\gamma)}(X,Z) = \varphi(X)e^{2\pi\mathbf{i}\langle\mathcal{Z}_\gamma,Z\rangle} = \varphi(X)e^{2\mathbf{i}\langle Z_\gamma,Z\rangle}. \tag{82}$$

(Lattice $\pi\Gamma_Z$ is introduced, mostly, for simplifying the formulas.) Note that (82) defines Γ_Z-periodic functions. Thus, this problem is ultimately reduced to computing the spectrum of the Zeeman Laplacian $\Box_{(\gamma)}$ introduced in (48). The computations are carried out by the second form of this operator, where the π is involved in the norm of Z_γ ($|Z_\gamma| = \pi|\mathcal{Z}_\gamma|$). Since the Hamiltonian $H_{Z_\gamma f} = -\frac{1}{2}\Box_\gamma$ differs from the classical Zeeman-Hamilton operator H_{Z_γ} just by an additive constant, this spectrum computation readily provides the spectrum also of a free electron orbiting around the origin in a constant magnetic field.

The subspace corresponding to $\gamma = 0$ is nothing but the space of functions depending only on the X-variable. Thus the corresponding Laplacian is the Euclidean Laplacian Δ_X and the spectrum is the well known continuous spectrum there. On the other subspaces, where $\gamma \neq 0$ and the endomorphism J_γ is non-degenerated, the spectrum is discrete which can explicitly be computed. If the J_γ is degenerated, then the Laplacian is the direct sum of the Euclidean Laplacian, defined on the maximal subspace where J_γ is degenerated, and of the non-degenerated magnetic Laplacian defined on the complement subspace. Thus the spectrum is the product of the continuous Euclidean spectrum by the discrete spectrum of the non-degenerated operator. By writing the corresponding decomposition of the Fourier-Weierstrass function spaces in the form

$$FW^\gamma = FW_0^\gamma \odot FW_{ML}^\gamma, \tag{83}$$

one has the decomposition

$$FW = FW_0 \odot FW_{ML} = (\oplus FW_0^\gamma) \odot (\oplus FW_{ML}^\gamma) \tag{84}$$

of the Γ_Z-periodic functions such that the spectrum is continuous on FW_0 (it is determined by the corresponding Euclidean spectra) and it is discrete on FW_{ML}. Note that on H-type groups the subspace FW_0 is nothing but FW^0 and FW_{ML} is spanned by all FW^γ where $\gamma \neq 0$. In what follows the spectrum is computed on FW_{ML} what is called *magnetic spectrum*, or, *Zeeman spectrum*. It simply means that J_γ is non-degenerated in the Laplacian (48).

Despite the discreteness, this spectrum is very different from those defined on compact manifolds. For instance, each eigenvalue has infinite multiplicity. Roughly speaking, this is one of the reasons for infinities appear in free electron theory. This paper approaches to this problem by decomposing the corresponding L^2-space into *Zeeman-zones* on which these multiplicities together with other important quantities become finite ones.

The differential operator (48) is built up by the following two commuting operators

$$\mathbf{O} = \Delta_X - 4(|Z_\gamma|^2 + \frac{1}{4}|J_{Z_\gamma}(X)|^2)\,,\ \mathbf{D} = 2\mathbf{i}D_{(\gamma)}\bullet\,. \tag{85}$$

Apart from the constant term $4|Z_\gamma|^2$, the first operator can be interpreted as the Schrödinger wave operator of a harmonic oscillator. The textbook solution of the eigenvalue problem of this operator uses Hermitian polynomials. We adopt this technique to the spectral computations developed in this paper. It should be mentioned, however, that these functions provide eigenfunctions only for the first operator above. In order to get eigenfunctions also with respect to the second operator (which commutes with the first one!), a certain decomposition technique on Hermitian polynomials shall be applied.

This computation technique can be described, in short, as *splitting the spectral lines of the harmonic oscillator operator* $\mathbf{O}$ *with the angular momentum operator* $\mathbf{D}$ *(Zeeman effect).* There is explained later that this is indeed different from the standard method, which neglects the oscillator potential, retains the Coulomb potential and the spectrum is computed by splitting the spectrum of the Coulomb-Schrödinger operator by $\mathbf{D}$. It turns out that the latter standard method is "blindfolded" regarding the Zeeman zones, meaning that the zones can not be constructed by it.

Eigenfunctions of the harmonic oscillator operator O. The standard form of the Hamilton operator for a harmonic oscillator is $\Delta_X - |X|^2$, which acts on $L^2(\mathbf{R}^k)$ by the Friedrichs extension. The 1-dimensional version of this operator is $S = d^2/dx^2 - x^2$, whose eigenvalues can be computed as

follows [Ta]. (We need this standard computations later, for establishing the spectrum of the complete operator $\Box_\gamma$.)

By introducing the first order operators

$$s^+ = \frac{d}{dx} + x \quad ; \quad s^- = \frac{d}{dx} - x \,, \tag{86}$$

we have

$$S = s^+ s^- + 1 = s^- s^+ - 1 \ , \ [s^+, s^-] = s^+ s^- - s^- s^+ = -2 \,. \tag{87}$$

Since $s^+(e^{-x^2/2}) = 0$, the L^2-eigenfunctions of S are the functions

$$(\frac{d}{dx} - x)^l e^{-x^2/2} = H_l(x) e^{-x^2/2} \,, \tag{88}$$

where $H_l(x)$ are the so called Hermite polynomials, given by

$$H_l(x) = (-1)^r e^{x^2} (\frac{d}{dx})^l e^{-x^2} = \sum_{j=0}^{[l/2]} (-1)^j \frac{l!}{j!(l-2j)!} (2x)^{l-2j} \,. \tag{89}$$

The computations below, implemented in the general cases, show that the eigenvalue of S, corresponding to the function (92), is $-(2l+1)$. Since the functions

$$c_l H_l(x) e^{-x^2/2} \ ; \ c_l = (\pi^{1/2} 2^l (l!))^{-1/2} \ ; \ l = 0, 1, \ldots , \tag{90}$$

form an orthonormal basis on $L^2(\mathbf{R})$, the eigenvalue problem of S is solved in the 1-dimensional case.

On $\mathbf{R}^k$ the eigeinfunctions of $S = \Delta_X - |X|^2$ are

$$h_{l_1 \ldots l_k}(X) = H_{l_1}(X^1) \ldots H_{l_k}(X^k) e^{-|X|^2/2} \ ; \ l = l_1 + \ldots + l_k \tag{91}$$

with eigenvalues $-(2l+k)$. In this case the functions $c_{l_1} \ldots c_{l_k} h_{l_1 \ldots l_k}(X)$ form an orthonormal basis in $L^2(\mathbf{R}^k)$.

Since the function

$$\langle -J^2_{Z_\gamma}(X), X \rangle = \langle K(X), X \rangle \tag{92}$$

differs from $|X|^2$ in general, one should modify the above formulas, since the corresponding spectrum, defined in (91), depends also on the eigenvalues of the positive self-adjoint operator K. In fact, let $\sqrt{K}$ be the uniquely

determined positive square root of the operator K. If $\sqrt{K}$ has a single eigenvalue λ then the corresponding eigenfunctions are

$$h_{\lambda,l_1,\ldots,l_k}(X) = H_{l_1}(\lambda X^1)\ldots H_{l_k}(\lambda X^k)e^{-\lambda|X|^2/2} \tag{93}$$

with eigenvalues $-(2l+k)\lambda$. The multiplicity of this eigenvalue is $\binom{l+k-1}{k-1}$, since this is the number of options for l can be expressed in the form $l = l_1 + l_2 + \ldots + l_k$, where l_i's are non-negative integers. (A reformulation of this combinatorial question is this: In how many ways can be l identically looking balls painted by k different colors?)

If $\sqrt{K}$ has the distinct eigenvalues $0 \leq \lambda_1 < \ldots < \lambda_s$ with multiplicities $k_1, \ldots, k_s$ on the corresponding k_i-dimensional subspaces $\mathcal{X}_i$ then the eigenfunctions have the following product form

$$\prod_{i=1}^{s} H_{r_{i1}}(\lambda_i X_i^1)\ldots H_{r_{ik_i}}(\lambda_i X_i^{k_i})e^{-\lambda_i|X_i|^2/2}, \tag{94}$$

where $X_i = (X_i^1, \ldots, X_i^{k_i})$ is a vector from $\mathcal{X}_i$ whose components are defined by an orthonormal basis on $\mathcal{X}_i$. The eigenvalues corresponding to these functions are

$$\sum_{i=1}^{s} -\lambda_i(k_i + 2\sum_{j=1}^{k_i} l_{ij}). \tag{95}$$

The multiplicity of this eigenvalue is the product,

$$\prod_{i=1}^{m} \binom{k_i - 1 + \sum_{j=1}^{k_i} l_{ij}}{k_i - 1}, \tag{96}$$

of the corresponding multiplicities determined on the eigensubspaces of K.

Eigenfunctions of the complete Laplacian $\Box_\gamma$. Functions (93) and (94) are not yet the eigenfunctions of the operator $\mathbf{i}D_\gamma\bullet$, which is also included into the complete magnetic Laplacian (48). Since this operator commutes with $\mathbf{O}$, one can generate a subspace which is invariant with respect to the complete operator by letting the operator $D_\gamma\bullet$ successively acting on functions (94). Next we describe this generated subspace which shall be decomposed into the direct sum of subspaces which are invariant with respect to the action of the magnetic Laplacian. Exactly this step in the computations is called *splitting the spectrum of* $\mathbf{O}$ *with the angular momentum operator.*

The main idea in these computations is that the linear functions $\langle Q + \mathbf{i}J(Q), X\rangle$ resp. $\langle Q - \mathbf{i}J(Q), X\rangle$, where the unit skew endomorphism J satisfying $J^2 = -id$ is defined by normalizing J_γ, are eigenfunctions of the

corresponding operator $D\bullet$, defined by J, with eigenvalue $\mathbf{i}$ resp. $-\mathbf{i}$. Thus the eigenvalue of $D\bullet$ on the l^{th} order polynomial $P_+^{(p)}(X)P_-^{(l-p)}(X)$, where

$$\begin{aligned} P_+^{(p)}(X) &= \langle Q_1^+ + \mathbf{i}J(Q_1^+), X\rangle \dots \langle Q_p^+ + \mathbf{i}J(Q_p^+), X\rangle, \\ P_-^{(l-p)}(X) &= \langle Q_1^- - \mathbf{i}J(Q_1^-), X\rangle \dots \langle Q_{l-p}^- - \mathbf{i}J(Q_{l-p}^-), X\rangle \end{aligned} \tag{97}$$

is $(p-(l-p))\mathbf{i} = (2p-l)\mathbf{i}$. This formula, where $p = 0, \dots, l$, describes the complete set of eigenvalues of $D\bullet$ on the l^{th} order polynomials.

First we deal with the case when $\sqrt{K}$ has only a single eigenvalue and, thus, the eigenfunction of $\mathbf{O}$ has the form (93). Let $(E_1, \dots, E_k)$ be the orthonormal basis on the X-space defining the coordinates $(X^1, \dots, X^k)$ and write each coordinate in the form

$$X^i = \frac{1}{2}\langle E_i + \mathbf{i}J(E_i), X\rangle + \frac{1}{2}\langle E_i - \mathbf{i}J(E_i), X\rangle. \tag{98}$$

Plug in these formulas in (93) and perform all of the powering and multiplications indicated in the Hermite polynomials. Then the polynomial $H_{l_1}(\lambda X^1)\dots H_{l_k}(\lambda X^k)$ becomes a linear combination of polynomials of the form (97).

For a fixed m, the function $H^{(m)}_{\lambda,l_1,\dots,l_k}(X)$ is defined by the sum of all those terms for which $2p-l = m$ holds. Note that these functions always contain an l^{th}-order term and, except for the functions $H^{(l)}_{\lambda,l_1,\dots,l_k}$ and $H^{(-l)}_{\lambda,l_1,\dots l_k}$, they always include also lower order terms. In other words, these polynomials are inhomogeneous and, therefore, they are different from the spherical harmonics used for similar spectral computations in general. (See more about this question in the last section of this chapter.) The complete range of possible m's is described by $2p-l$, where $p = 0, \dots, l$. Such a function is an eigenfunction of the operator $\mathbf{D} = \mathbf{i}D\bullet$ with eigenvalue $-(2p-l)\lambda$. Therefore, by letting $\mathbf{D}$ successively acting on

$$H_{\lambda,l_1,\dots,l_k}(X) = \sum_{p=0}^{l} H^{(2p-l)}_{\lambda,l_1,\dots,l_k}(X), \tag{99}$$

we have

$$\mathbf{D}^i(H_{\lambda,l_1,\dots,l_k})(X) = \sum_{p=0}^{l}(-(2p-l)\lambda)^i H^{(2p-l)}_{\lambda,l_1,\dots,l_k}(X). \tag{100}$$

Since $\mathbf{D}e^{-\lambda|X|^2/2} = 0$, thus (99) and (100) hold also for the functions

$$h_{\lambda,l_1,\dots,l_k}(X)\ ,\ h^{(2p-l)}_{\lambda,l_1,\dots,l_k}(X) = H^{(2p-l)}_{\lambda,l_1,\dots,l_k}(X)e^{-\lambda|X|^2/2}. \tag{101}$$

Note that equations (100), considered for the values $i = 0, \ldots, p$, allow to express the functions $H^{(2p-l)}_{\lambda,l_1,\ldots,l_k}$ via the functions $\mathbf{D}^i H_{\lambda,l_1,\ldots,l_k}$. In fact, these equations establish a linear connection between the two fields of $(l+1)$-vectors whose coordinate functions are the above functions. The matrix of this linear transform is non-degenerated. In fact, if l is an odd number then it is a Vandermonde matrix, V, with entries $V_{ip} = (-(2p-l)\lambda)^i \neq 0$. If l is even, the first row consists of 1's and except the first one all the elements are 0's in the middle column. The rest part is again a Vandermonde matrix. In either cases the V is invertible and

$$H^{(2p-l)}_{\lambda,l_1,\ldots,l_k} = \sum_{i=0}^{r} V^{-1}_{pi} \mathbf{D}^i H_{\lambda,l_1,\ldots,l_k}. \tag{102}$$

The same equations hold corresponding to the functions (101). By the commutativity of operators $\mathbf{O} = \Delta_X - 4(|Z_\gamma|^2 + \frac{1}{4}|J_{Z_\gamma}(X)|^2)$ and $2\mathbf{D}$ we get that the functions

$$h^{(2p-l)}_{\lambda,l_1,\ldots,l_k}(X) = H^{(2p-l)}_{\lambda,l_1,\ldots,l_k}(X) e^{-\lambda|X|^2/2} \tag{103}$$

are eigenfunctions of the complete Zeeman-Laplace operator with eigenvalue

$$-((4p+k)\lambda + 4k\lambda^2). \tag{104}$$

The first term was computed as the sum of $-(2l+k)\lambda$ and $-2(2p-l)\lambda$.

Next the multiplicities of eigenvalues are determined for a fixed value l. One should bear in mind this restriction in these computations. The multiplicities on the whole $L^2(\mathbf{v})$ will be discussed later.

For a fixed l the computation of multiplicities can be established by noticing that the spectrum-elements (104) depend only on the holomorphic parameter p and they do not depend on the anti-holomorphic parameter $\upsilon = l - p$. Therefore, the complex valued l^{th}-order polynomial space splits into $(l+1)$ number of invariant subspaces determined by the values $p = 0, 1, \ldots, l$. For a fixed p the complex dimension of the corresponding subspace is

$$\binom{p + k/2 - 1}{k/2 - 1} \binom{\upsilon + k/2 - 1}{k/2 - 1}, \tag{105}$$

which number is the multiplicity of the eigenvalue corresponding to p. (This multiplicity is computed on the complex space $\mathbf{C}^{k/2}$. The first term in (105) is the number of decompositions $p = l_1 + \ldots + l_{k/2}$. The second factor has the same meaning corresponding to υ.) This shows that the anti-holomorphic

parameter, $\upsilon = l - p$, plays role only in the multiplicity of a spectrum-element.

When $\sqrt{K}$ has distinct eigenvalues, $\lambda_i\,;\, i = 1, \ldots, s$, then the eigenfunctions are the products of functions

$$h^{(2p_i - \sum_j l_{ij})}_{\lambda_i, l_{i1}, \ldots, l_{ik_i}}(X) = H^{(2p_i - \sum_j l_{ij})}_{\lambda_i, l_{i1}, \ldots, l_{ik_i}}(X) e^{-\lambda_i |X_i|^2/2}, \tag{106}$$

where X_i represents a generic vector from the eigensubspace $\mathcal{X}_i$ belonging to the eigenvalue λ_i. Thus we have

Theorem 4.1 (Explicit Spectral Theorem) *Laplacian Δ_N leaves the function spaces $FW^{(\gamma)}$, defined for the lattice points $\mathcal{Z}_\gamma \in \Gamma$ by functions (82), invariant and it acts on such a subspace as the magnetic Laplacian (48). This latter operator is the sum of the harmonic oscillator operator $\mathbf{O}$ and of the angular momentum operator $\mathbf{D}_{(\gamma)}$. These two operators commute.*

For $\gamma = 0$, the Laplacian $\Box_{(0)}$ is nothing but the Euclidean Laplacian Δ_X which has the well known continuous spectrum on $FW^{(0)}$. For a $\gamma \neq 0$, where the J_γ is non-degenerated, the spectrum is discrete on $FW^{(\gamma)}$. More precisely, the eigenfunctions can be expressed in the product form

$$\prod_{i=1}^{m} h^{(2p_i - \sum_j l_{ij})}_{\lambda_{\gamma i}, l_{i1}, \ldots, l_{ik_i}}(X_i) \tag{107}$$

which have the corresponding eigenvalues

$$\sum_{i=1}^{s} -(\lambda_{\gamma i}(4p_i + k_i) + 4\lambda_{\gamma i}^2 k_i). \tag{108}$$

For fixed values p_i , $\upsilon_i = \sum_j l_{ij} - p_i$, where $i = 1, 2, \ldots, s$, the multiplicity (the complex dimension of the corresponding eigensubspace) of such an eigenvalue is

$$\prod_{i=1}^{s} \binom{p_i + k_i/2 - 1}{k_i/2 - 1} \binom{\upsilon_i + k_i/2 - 1}{k_i/2 - 1}. \tag{109}$$

If the J_γ is degenerated, the spectrum is continuous which is, formally, the product of the continuous spectrum corresponding to the subspace where J_γ is degenerated and of the discrete spectrum corresponding to the subspace where J_γ is non-degenerated and where the spectrum can be computed by the above method. On Heisenberg-type groups the only subspace on which the spectrum is continuous is $FW^{(0)}$ and the spectrum is discrete on the other subspaces.

Zones defined spectrally. The gross Zeeman zones were defined, in Section 2, as the semi-irreducible invariant subspaces of the complex Heisenberg group representation. There is also described the number of irreducible zones a gross zone decomposes into. In this section the zones are defined by means of the explicit spectrum established above. If there is only a single parameter $\lambda \neq 0$ involved, this decomposition is defined, in this new way, by the eigensubspaces spanned by functions $H^{(m)}_{\lambda,\mathbf{l}}$, defined in (99). In the following these subspaces are denoted also by $\mathbf{H}^{(p,l-p)}$. These polynomials are constructed by the l^{th}-order polynomials $P^{(p,l-p)}$, where p resp. $(l-p)$ are the holomorphic resp. antiholomorphic degrees, by a splitting technique using Vandermonde matrices. The orders p and $\upsilon = l - p$ are called *holomorphic* and *antiholomorphic indexes* also with respect to the eigenfunctions. The connection between the pairs (m, l) and $(p, l-p)$ can be established by the formula $m = 2p - l$. The elements of the spectrum are $(4p + k)\lambda + 4k\lambda^2$.

When more λ's, $(\lambda_1, \ldots, \lambda_s)$, are involved the corresponding subspaces are denoted by

$$\mathbf{H}^{(p_1,\ldots,p_s,\upsilon_1,\ldots,\upsilon_s)} \quad , \quad \upsilon_i = l_i - p_i, \tag{110}$$

which can be considered as the tensor product of the function spaces $\mathbf{H}^{(p_i,\upsilon_i)}_i$.

By formulas (104) and (108), the actual eigenvalues do not depend on the antiholomorphic indexes, therefore, each eigenvalue is constant on the infinite dimensional function space

$$IM^{(p_1,\ldots,p_s)} = \sum_{(\upsilon_1,\ldots,\upsilon_s)} \mathbf{H}^{(p_1,\ldots,p_s,\upsilon_1,\ldots,\upsilon_s)} e^{-\frac{1}{2}\sum \lambda_i |X_i|^2}. \tag{111}$$

In other words, each eigenvalue has infinite multiplicity.

For fixed antiholomorphic degrees $(\upsilon_1, \ldots, \upsilon_s)$ consider the direct sum:

$$FSGZ^{(\upsilon_1,\ldots,\upsilon_s)} = \sum_{(p_1,\ldots,p_s)} \mathbf{H}^{(p_1,\ldots,p_s,\upsilon_1,\ldots,\upsilon_s)} e^{-\frac{1}{2}\sum \lambda_i |X_i|^2}, \tag{112}$$

which defines a so called Fine Spectral Gross Zone corresponding to the Fine Zone Indexes $FZI = (\upsilon_1, \ldots, \upsilon_s)$. The Spectral Gross Zone, SGZ^{υ}, corresponding to the index $GZI = \upsilon = \upsilon_1 + \ldots + \upsilon_s$ is the direct sum

$$SGZ^{\upsilon} = \sum_{\upsilon=\upsilon_1+\ldots+\upsilon_s} FSGZ^{(\upsilon_1,\ldots,\upsilon_s)}. \tag{113}$$

In order to define the Irreducible Spectral Zones, consider a complex coordinate neighborhood $z_1 \ldots, z_{k/2}$ of the complex structure J defined by normalizing J_γ such that the J_γ has the eigenvalue λ_i on a complex plane z_i.

If $p_i, l_i, u_i = l_i - p_i$ are given for all complex plane z_i, then the Irreducible Spectral Zone $ISZ^{(u_1,\ldots,u_{k/2})}$ regarding the index $(u_1,\ldots,u_{k/2})$ is defined in the same way as the fine zone (112).

For an eigenfunction the *gross holomorphic, antiholomorphic, zonal azimuthal, and magnetic indexes (quantum numbers)* are defined by $p = \sum p_i$, $u = \sum u_i, l = \sum l_i$ and $m = 2p - l$ respectively. For a fixed complex plane z_i the corresponding indexes are called individual indexes. These names indicate that each coordinate represent an individual charged particle. Then we have:

Theorem 4.2 (Splitting Theorem) *The gross spectral zones are invariant under the action of the magnetic Laplacian (48). If $\lambda_i \neq 0$, for all i, the spectrum restricted to a gross zone is discrete such that each element has finite multiplicity.*

The spectrum-elements on two different gross zones are the same. The multiplicities, described in (109) for fixed fine zone index $FZI = (\upsilon_1,\ldots,\upsilon_s)$, are different, however. Two spectra yielding this property are called isochromatic. If $k = 2$, any two gross zones are isospectral. If $k > 2$, any two distinct gross zones are properly isochromatic having distinct multiplicities for the same eigenvalues.

By the formula $\frac{1}{2}(l - m) = u$, the gross zonal azimuthal number and the magnetic number of the eigenfunctions uniquely determine the gross zone they belong to.

Each spectral gross zone decomposes into $\binom{\upsilon+(k/2)-1}{\upsilon} = \binom{\upsilon+(k/2)-1}{(k/2)-1}$ number of irreducible spectral zones, $ISZ^{(u_1,\ldots,u_{k/2})}$. The irreducible zone to which an eigenfunction belongs to is uniquely determined by the individual magnetic and zonal azimuthal numbers of the eigenfunction. The multiplicity of each eigenvalue on an irreducible zone is 1 if and only if the eigenvalues λ_i are distinct. Thus higher zonal multiplicities indicate the presence of particles which are identical up to the sign of the charge. Any two irreducible spectral zones are isospectral.

Finally, the identity of the earlier introduced and the spectral zones is established.

Theorem 4.3 *The spectral gross zones are semi-irreducible invariant subspaces under the action of the complex Heisenberg algebra representation*

$$\rho_{\mathbf{c}}(z^i)(\psi) = (-\partial_{\overline{z}^i} + \lambda_i z^i \cdot)\psi \quad , \quad \rho_{\mathbf{c}}(\overline{z}^i)(\psi) = \partial_{z^i}\psi \tag{114}$$

defined in (35) on the complex Hilbert space $L^2_{\mathbf{C}\eta_{\lambda_i}}$, where $\eta_{\lambda_i} = e^{-\sum \lambda_i z^i \overline{z}^i}$.

Actually, the spectral zones (gross and irreducible) are nothing but the ones defined earlier by the extended Fock representation.

Proof. For a fixed zonal azimuthal quantum number l, the spectral Zeeman zones are distinguished by the distinct eigenvalues $m = 2p - l$ of the skew operator $\mathbf{D}$. Therefore two distinct zones are perpendicular to each other. Operators ∂_{z_i} act, by definition, on function spaces $\mathbf{H}_\gamma^{(p_1,\ldots,p_s,\upsilon_1,\ldots,\upsilon_s)}$ and thus the spectral Zeeman zones are invariant under these actions. Since the operators $-\partial_{\overline{z}_i} + \lambda_i z_i \cdot$ and ∂_{z_i} are dual, the invariance with respect to both operators follows by the above orthogonal property.

The first spectral zone is obviously $\mathcal{H}^{(0)}$. The second one is in $G^{(0)} \oplus G^{(1)}$ such that it is perpendicular to $G^{(0)}$. Therefore, it must be $\mathcal{H}^{(1)}$. Mathematical induction proves that the zones introduced by the extended Fock representation are the same as the spectral zones. □

Standard computation of the Zeeman spectrum. In order to establish a precise connection between the above and classical quantum numbers of particle theory, in this section the spectral computations are accomplished in the standard way they are carried out in classical quantum mechanic for computing the spectrum of an electron. These classical computations always involve the Coulomb potential of the nucleus. It should be noted that the zones are not invariant with respect to multiplications with radial functions, thus the technique establishing the Zeeman zones can not be applied to the Coulomb-Schrödinger operator. The only method for computing the spectrum in this case is the one which traces back the eigenvalue problem to the radial functions $f(\langle X, X\rangle)$ by seeking the eigenfunctions in the form fH, where the H is a homogeneous harmonic polynomial.

This method is really different from the technique of splitting the spectrum of $\mathbf{O}$ with $\mathbf{D}_\gamma$, by which the Zeeman zones are constructed. On the complex plane, for instance, the homogeneous harmonic polynomials can be divided into two classes. One of them contains the holomorphic polynomials, the other the antiholomorphic ones. By this classification the zones can not be discovered. Probably this is why the Zeeman zones were not recognized in the literature earlier.

This standard method is a variant of the technique of *splitting the spectrum of the Coulomb-Schrödinger operator with the angular momentum operator,* by which the Zeeman effect (quantization of magnetic dipole moment) was originally explained. The word "variant" is justified, since the standard computations usually neglect the quadratic oscillator potential and retain the Coulomb potential V, while in our case $V = 0$, however, both the

harmonic oscillator potential and the field energy is retained.

The computations will be carried out for such a Zeeman operator H_{Zf} which depends only on a single parameter λ. The eigenfunctions of this operator are sought now in the form $F(X) = f(\lambda\langle X, X\rangle)H^{(\tilde{l},m)}(X)$, where f is a real valued smooth even function defined on $\mathbf{R}$ and $H^{(\tilde{l},m)}(X)$ is a complex valued homogeneous harmonic polynomial of order $\tilde{l}$ on the X-space such that it is eigenfunction also of the operator $D\bullet$, introduced by means of the unit endomorphism J in §2, with eigenvalue $m = 2\tilde{p} - \tilde{l}$. When more parameters, $\{\lambda_i\}$, are involved, the eigenfunctions are represented as product of functions of the form $F_{(i)}(X) = f_{(i)}(\lambda_i\langle X, X\rangle)H_{(i)}^{(\tilde{l}_i,m_i)}(X)$, where the functions in the formula are defined on the maximal eigensubspace corresponding to the parameter λ_i.

One can trace back the eigenvalue problem of $\Box_\lambda$ to the eigenvalue problem of an ordinary differential operator acting on the radial functions $f(\langle X, X\rangle)$ as follows. First the simplest case is described where $\lambda = 1$. Since $D_\lambda \bullet f = 0$, $|Z_\lambda|^2 = k$, and $|J_\lambda(X)|^2 = \langle X, X\rangle$, we get

$$(\Box_{(\lambda)}F)(X) = (4\langle X, X\rangle f''(\langle X, X\rangle) + (2k + 4\tilde{l})f'(\langle X, X\rangle) \tag{115}$$

$$-(2m + 4(k + \frac{1}{4}\langle X, X\rangle)f(\langle X, X\rangle)))H^{(\tilde{l},m)}(X). \tag{116}$$

The eigenvalue problem is reduced, therefore, to the ordinary differential operator

$$(L_{(\lambda=1,l,m)}f)(t) = 4tf''(t) + (2k + 4\tilde{l})f'(t) - (2m + 4(k + \frac{1}{4}t))f(t). \tag{117}$$

The function $e^{-\frac{1}{2}t}$ is an eigenfunction of this operator with eigenvalue $-(4\tilde{p} + 5k)$. The general eigenfunctions are sought in the form

$$f(t) = u(t)e^{-\frac{1}{2}t}. \tag{118}$$

Such a function is an eigenfunction of $L_{\tilde{l},m}$ if and only if the $u(t)$ is an eigenfunction of the differential operator

$$(P_{(\lambda=1,\tilde{l},m)}u)(t) = 4tu''(t) + (2k + 4\tilde{l} - 4t)u'(t) - (4\tilde{p} + 5k)u(t). \tag{119}$$

The most remarkable property of this operator is that it has a uniquely determined polynomial eigenfunction

$$u_{(\lambda=1,n,\tilde{l},m)}(t) = t^n + a_1t^{n-1} + a_2t^{n-2} + \ldots + a_{n-1}t + a_n \tag{120}$$

with coefficients satisfying the recursion formulas

$$a_0 = 1 \quad , \quad a_i = -a_{i-1}(n-i)(n+\tilde{l}+\frac{1}{2}k+1-i)n^{-1}. \tag{121}$$

One can easily establish explicit combinatorial formula for a_i by this recursion, yet we do not deal with this problem here. The eigenvalue corresponding to this polynomial is

$$\mu_{(\lambda=1,n,\tilde{l},\nu)} = -(4n+4p+5k) \quad , \quad p = \frac{1}{2}(m+\tilde{l}). \tag{122}$$

The polynomials (120) are nothing but the Laguerre polynomials. It was mentioned earlier that they are defined as the n^{th}-order polynomial eigenfunctions of the differential operator

$$\Lambda_\alpha(u)(t) = tu'' + (\alpha+1-t)u', \tag{123}$$

with eigenvalues $-n$. Therefore

$$P_{(\lambda=1,\tilde{l},m)} = 4\Lambda_{(\frac{1}{2}k+\tilde{l}-1)} - (4p+5k). \tag{124}$$

Thus the eigenfunctions of operators (119) and (123) are the same. The relations between the corresponding eigenvalues are properly described in (122). Particularly we get that, for fixed values for $k, \tilde{l}, m$ and thus also for p, the functions $u_{(\lambda=1,n,\tilde{l},m)}$, $n = 0, 1, \ldots \infty$ form a basis in $L^2([0,\infty))$.

In case of a single λ, differential operator (117) is of the form

$$(L_{(\lambda,\tilde{l},m)}f)(t) = 4tf''(t) + (2k+4\tilde{l})f'(t) - (2m\lambda + 4\lambda^2(k+\frac{1}{4}t))f(t). \tag{125}$$

The eigenfunctions are sought in the form

$$u(\langle X,X\rangle)e^{-\frac{1}{2}\lambda\langle X,X\rangle}\Phi_{(\tilde{l},m)}(X), \tag{126}$$

which leads to the radial operator

$$(P_{(\lambda,\tilde{l},m)}u)(t) = 4tu''(t) + (2k+4\tilde{l}-4\lambda t)u'(t) - (2m\lambda+4k\lambda^2)u(t), \tag{127}$$

corresponding to (119). The polynomial eigenfunctions relate to the above Laguerre polynomials by the simple formula

$$u_{(\lambda,n,\tilde{l},m)}(t) = u_{(\lambda=1,n,\tilde{l},m)}(\lambda t). \tag{128}$$

Then the eigenvalue regarding the corresponding eigenfunction of (127) is

$$\mu_{(\lambda,n,\tilde{l},m)} = -((4n + 4\tilde{p} + k)\lambda + 4k\lambda^2). \tag{129}$$

These actual eigenvalues depend on $p = n+\tilde{p}$. If this parameter is fixed, the multiplicity of the corresponding eigenvalue is

$$\sum_{n=0}^{p} \binom{p-n+k-1}{k-1} \binom{\tilde{v}+k-1}{k-1}, \tag{130}$$

Finally, if there are distinct eigenvalues $\{\lambda_1, \ldots, \lambda_s\}$ involved, the eigenfunctions are the products of functions of the form

$$u_{(\lambda=1,\tilde{l},m,n)}(\lambda_i \langle X, X\rangle_i) e^{-\frac{1}{2}\lambda_i \langle X,X\rangle_i} \Phi_{(\tilde{l}_i,m_i)}(X_i), \tag{131}$$

where X_i is the orthogonal projection of a vector X onto the maximal eigensubspace belonging to λ_i and $\langle X, X\rangle_i = \langle X_i, X_i\rangle$. Then the eigenvalue corresponding to this function is

$$\mu_{(\lambda,n,\tilde{l},m)} = -\sum_{i=1}^{s}((4n_i + 4\tilde{p}_i + k_i)\lambda_i + 4k_i\lambda_i^2), \tag{132}$$

where $n = \sum n_i$, $l = \sum \tilde{l}_i$, $m = \sum m_i$, and $\tilde{v}_i = \tilde{l}_i - \tilde{p}_i$.

The multiplicity of the eigenvalue corresponding to fixed values of parameters $p_i = n_i + \tilde{p}_i$ are

$$\prod_{i=1}^{s} \sum_{n_i=0}^{p_i} \binom{p_i - n_i + k_i - 1}{k_i - 1} \binom{\tilde{v}_i + k_i - 1}{k_i - 1}. \tag{133}$$

Zeeman effect. According to the hypothesis, an electron of charge $-e$ and mass M revolving about the origin (nucleus) has *magnetic dipole moment* $\mathcal{S} = -(eh/2Mc)\mathcal{L}$ associated with the *angular moment* $J_x = hL_x = yp_z - zp_y, \ldots$ etc.. The influence of $\mathcal{L}$ must be clearly felt under magnetic influence. This influence was measured by the Stern-Gerlach experiment such that a stream of one electron atoms was moved in the x-direction through a constant magnetic field $\mathbf{K} = B\partial_z$ pointing into the z-direction. In performing the experiment on silver atoms in normal state the beam was broken up into 2 beams (Zeeman effect) proving the reality of the magnetic dipole moment.

In the mathematical model of this paper the angular momentum, (J_x, J_y, J_z), corresponds to the endomorphisms J_γ and the quantum angular momentum operator is $2\mathbf{D}_\gamma \bullet$. The Zeeman effect can be interpreted as the

influence this operator exerts on the spectrum of the associated operators such as the harmonic oscillator operator $\mathbf{O}$ or the Coulomb operator. Since it commutes with both of them, the angular momentum operator breaks up (splits) the spectral lines of both associated operators, defining finer spectra. This effect is exhibited in the classical splitting

$$\mathbf{H}^{(\tilde{l})}e^{-\frac{1}{2}\lambda|X|^2} = \oplus_p \mathbf{H}^{(p,\tilde{l}-p)}e^{-\frac{1}{2}\lambda|X|^2} \tag{134}$$

as well as in the Zeeman zone decomposition.

In classical quantum theory the quantum numbers are established by means of this standard spectral computation. The quantum angular momentum operator has the eigenvalues $m = 2\tilde{p} - \tilde{l}$ on such a common eigensubspace $\mathbf{H}^{(\tilde{l})}$ and the range of these eigenvalues is $m = -\tilde{l}, \ldots \tilde{l}$. Values $\tilde{l}$ and m are called *azimuthal quantum numbers* and *magnetic quantum numbers* respectively. The quantum physical interpretation of these numbers is the following: The angular momentum is quantized and in a classical azimuthal quantum state, $\tilde{l}$, it can take $\tilde{l}+1$ distinct values determined by the formula $m = 2\tilde{p} - \tilde{l}$, where p runs through the values $\tilde{p} = 0, \ldots, \tilde{l}$. Thus a beam of electrons being on the same azimuthal quantum state must split up into $\tilde{l}+1$ smaller beams. Comparing the zonal and classical quantum numbers note that the corresponding azimuthal numbers are different while the magnetic numbers are the same.

This was the very first explanation given for the Zeeman effect. In this interpretation only orbital angular momentum is attributed to the charged particle. Because of the discrepancies between the computed and measured quantities, Pauli added a non-relativistic spin angular momenta to the orbital one. Dirac developed the relativistic version of this operator. These operators are considered in separate papers [Sz5, Sz6]. The corresponding zones are called *anomalous zones*.

Due to the irreducibility, in quantum physics mostly the holomorphic zone, described by $\upsilon = 0$, is involved to theoretical explanations. One of the new features in this paper is that the quantization developed here applies reducible Heisenberg group representation (extended Fock representation). This reducibility is inevitable since the quantization of the angular momenta can not be solved on a single zone. Indeed, for a fixed zonal azimuthal quantum number l, every zone represents only a single quantum number $m = 2p - l$. In other words, without using the reducible representation the quantization of the angular momentum could have not been established.

In the original approach, where the Zeeman operator is defined on $\mathbf{R}^3$, the 3-space is endowed by the polar (spherical) coordinate system (x, y, r)

and the z-coordinate is eliminated by the equation $x^2+y^2+z^2=r^2$. Then the eigenfunctions of the Hamilton operator are sought again in the form $F(r)h^{(\tilde{l})}(x,y)$, where the $h^{(\tilde{l})}$ is an $\tilde{l}^{th}$-order spherical harmonics whose degree defines the azimuthal quantum number. According to H. Weyl [We], pages 66, the silver atom can be in three normal states corresponding to $m=-1,0,+1$ associated with the functions $\overline{z}, z\overline{z}, z$. Then Weyl continues:" On performing the experiment on silver atoms in the normal state two beams, corresponding to $m=+1,-1$, were observed. Why the unperturbed beam corresponding to $m=0$ did not appear remained unexplained."

In establishing the Zeeman zones the eigenfunctions are sought not in the above form, due to that that the zones are not invariant with respect to multiplications by radial functions. A possible explanation to the above problem may be that that the functions $\overline{z}$ and $z\overline{z}$ are in the same zone, $\mathcal{H}^{(1)}$, thus one can talk about just two normal state in zonal theory.

Standard method used for isospectrality constructions. So far, the explicit spectrum has been computed on the non-compact manifolds $\Gamma_Z\backslash N$. If one considers compact sub-torus-bundles with boundaries (for instance, ball×torus-type domains which have sphere×torus-type boundaries), the technique described in the previous section can not be straightforwardly adopted for computing the spectrum. One of the advantages of determining the eigenfunctions in the standard way is that this technique can be applied also for these compact domains. This method was used in [Sz3, Sz4] for computing the spectrum both on ball×torus- and sphere×torus-type domains of Heisenberg-type Lie groups, in which case only a single eigenvalue λ is involved to the computations. Then, on ball×torus-type domains, one has to determine both the Dirichlet and Neumann spectra of operator $\Lambda_{(\frac{1}{2}k+\tilde{l}-1)}$ defined by formula (123) on the interval $[0,R]$, where R is the radius of the ball. Comparing with the non-compact case, there arise numerous differences. First of all, the eigenfunctions are not polynomials and they are not known explicitly. One can establish only implicit formulas, and as a result, one can not entirely explicitly compute the eigenvalues $\rho_{(k,\tilde{l},n)}\rightarrow-\infty$ of the Laguerre operator $\Lambda_{(\frac{1}{2}k+\tilde{l}-1)}$ with the given boundary conditions on a compact interval. The eigenvalues of the ball×torus-type domain depend on this implicitly determined spectrum by the semi-explicit formula

$$\mu_{(\lambda,\tilde{l},m,n)}=-((-4\rho_{(k,\tilde{l},n)}+4p+k)\lambda+4k\lambda^2). \tag{135}$$

For fixed n and $\tilde{l}$, the multiplicity of such an eigenvalue is the dimension of the space of the corresponding spherical harmonics $H^{(\tilde{l},m)}$. These computations are not used in this paper.

PART TWO

NORMAL DE BROGLIE GEOMETRY

Technical introduction. In the next chapters the zonal versions of the global heat, e^{-tH}, and Feynman-Dirac kernel, $e^{-\mathbf{i}tH}$, are established, where H stands for H_Z or $H_{Zf} = -\frac{1}{2}\Box_\gamma$. Remember that these two operators differ from each other just by the constant term $2|Z_\gamma|^2$. The global Wiener-Kac (WK) and Feynman-Dirac (FD) kernels defined by H_Z are denoted in the unified form

$$d_\sigma(t, X, Y) = e^{-\sigma t H_Z}(X, Y), \tag{136}$$

where $\sigma = 1$ resp. $\sigma = \mathbf{i}$ indicate the WK- resp. FD-kernel. For H_{Zf} these kernels will be denoted differently. The heat kernel is the fundamental solution of the heat equation by which the heat distribution $u_t(X)$ for a given initial function $u_0(X)$ can be constructed by the convolution $u_t(X) = \int d_1(t, X, Y)u_0(Y)dY$. Using FD-kernels, $d_\mathbf{i}$, in this formula, one has the solutions of the Schrödinger equation. (The computations are carried out only for these two values of σ, however, the formulas established later easily extend to kernels defined by arbitrary unit complex number σ.)

For a non-degenerated J_γ the spectrum, $\{\mu_i \geq 0\}$, is discrete, thus these kernels can be introduced by the infinite series

$$d_\sigma(t, X, Y) = \sum e^{-t\sigma\mu_i}\psi_i(X)\overline{\psi}_i(Y), \tag{137}$$

where the elements of the orthonormal basis $\{\psi_i\}$ on the L^2-space are eigenfunctions of operator H_Z. These kernels satisfy the limit-property

$$\lim_{t\to+0} d_\sigma(t, X, Y) = \delta_X(Y). \tag{138}$$

The kernels defined for H_{Zf} are denoted by $p_\sigma(t, X, Y)$. Because of the simple connection between the two kind of kernels, it is enough to consider only the classical kernels d_σ.

The existence of the global kernels seems to be jeopardized by the infinite multiplicities of the eigenvalues. Yet, surprisingly enough, (137) defines smooth kernels in either cases. They are explicitly computed in the next section. The existence of the heat kernel $p_1(t, X, Y)$ follows from that fact that operator $\Box$ is derived from the Laplacian of a Riemannian manifold. By this reason there exist also well defined Wiener-Kac measures, w_{xy}, on the path-spaces $\mathcal{P}_{xy}$, consisting of continuous curves connecting two arbitrary points x and y. The heat kernel is not of the trace class, however. Let it

be mentioned that by the technique of regularization one can define relative heat kernels of the trace class which define relative zeta and eta functions.

More serious problems arise with respect to the Dirac-Feynman kernel $d_{\mathbf{i}}(t,X,Y)$. By (150) this kernel is of the form $A(t)e^{B(t,X,Y)\mathbf{i}}$. Thus, for fixed t and X, the integral $\int d_{\mathbf{i}}(t,X,Y)dY$ does not exist (the integrand is not absolute integrable, anyway). Also the approximating measures $w^{(n)}_{\mathbf{i}xy}$, defined analogously to the approximating measures of the WK-measure, are divergent. It is well known from the history of this problem that Kac was able to define this measure only by using the heat kernel. (Regarding the Euclidean Laplacian Δ_X, this measure had been constructed by Wiener, earlier.) Later, Feynman and Kac established also the Radon-Nikodym derivative of the Wiener-Kac measure with respect to the Wiener measure. The corresponding formula became known as the Feynman-Kac formula.

Thus it is surprising that both zonal flows, introduced by restricting the global flows onto the zones, define trace class zonal kernels. Additionally, both define the corresponding zonal measures, $w^{(a)}_{\sigma}$, on the path-spaces $\mathcal{P}_{xy}$ rigorously.

5 Global WK- resp. DF-kernels

The explicit computation of the global heat kernel is greatly facilitated by the explicit kernel $D = e^{-\frac{1}{2}t(-\mathbf{i}\nabla-\mathbf{a})^2}$ established, in [AHS, Si], on $\mathbf{R}^3$, where the vector potential $a_1 = -(1/2)Bx_2, a_2 = \frac{1}{2}Bx_1, a_3 = 0$ corresponds to the constant magnetic field $(0,0,B)$. The other physical quantities such as $\hbar, e, c, M$ are chosen to be unit numbers. According to this result, the kernel in the question is as follows:

$$D(t,x_1,x_2,x_3) = \frac{B}{(2\pi t)^{1/2}4\pi sinh(\frac{1}{2}Bt)}exp(-\frac{1}{2t}(x_3-y_3)^2 \qquad (139)$$
$$-\frac{B}{4}coth(\frac{1}{2}Bt)[(x_2-y_2)^2+(x_1-y_1)^2]-\frac{1}{2}\mathbf{i}B(x_1y_2-x_2y_1)).$$

The computations are carried out by the Feynman-Kac-Ito formula regarding the well defined Wiener measures (Brownian motion technique). Since the Hamiltonian in this case is the sum of H_Z, defined on the (x_1,x_2)-plane, and $-\frac{1}{2}\partial^2_{x_3}$, the above heat flow is the product of the 1-dimensional Euclidean heat flow and

$$d_{1\gamma}(t,X,Y) = e^{-tH_0}(t,X,Y) = \qquad (140)$$
$$\frac{B/2}{2\pi sinh(\frac{1}{2}Bt)}e^{-\frac{B}{4}coth(\frac{1}{2}Bt)|X-Y|^2-\frac{1}{2}\mathbf{i}B\langle X,J(Y)\rangle}$$

$$= \frac{\lambda}{2\pi sinh(\lambda t)} e^{-\frac{\lambda}{2}coth(\lambda t)|X-Y|^2 - \mathbf{i}\langle X, J_\gamma(Y)\rangle},$$

where $\lambda = (1/2)B$. (Note that J_γ involves λ, since the eigenvalues of $-J_\gamma^2$ are λ^2.)

One has a self-contained proof for this formula, however, by showing that this kernel satisfies the heat equation $(\partial_t + H_{ZX})d_{1\gamma}(t, X, Y) = 0$ as well as the limit property (138). The heat equation can be readily established by the following computations, carried out for $k = 2$.

$$\partial_t(d_{1\gamma})(t, X, Y) = (\, -\lambda coth(\lambda t) + \frac{\lambda^2|X-Y|^2}{2sinh^2(\lambda t)})d_{1\gamma}(t, X, Y), \tag{141}$$

$$\begin{aligned} H_{ZX}(d_{1\gamma})(t, X, Y) = (\lambda coth(\lambda t) + \frac{\lambda^2}{2}\{-coth^2(\lambda t)|X-Y|^2 \\ +|Y|^2 - 2\mathbf{i}coth(\lambda t)\langle X, J(Y)\rangle - 2\mathbf{i}coth(\lambda t)\langle J(X), Y\rangle \\ -2\langle X, Y\rangle + |X|^2\})d_{1\gamma}(t, X, Y) = -\partial_t(d_{1\gamma})(t, X, Y). \end{aligned} \tag{142}$$

For a general k the kernel considered is the product of kernels determined for the coordinate planes z_i. Thus the heat equation is established in generality. Limit property (138) will be shown after introducing the zonal flows $d_1^{(a)}(t, X, Y)$ and, then, by proving that

$$\lim_{t\to+0} d_{1\gamma}^{(a)}(t, X, Y) = \delta_\gamma^{(a)}(X, Y), \tag{143}$$

valid for general k-dimensions.

The eigenvalues of the classical Zeeman-Hamilton operator and the one introduced in this paper differ from each other by the extra $2|Z_\gamma|^2 = 2\lambda^2$ constant. Thus, for $k = 2$, the sought heat kernel is

$$p_{1\gamma}(t, X, Y) = e^{-2t|Z_\gamma|^2} d_{1\gamma}(t, X, Y). \tag{144}$$

The heat kernel on the torus bundle on $\Gamma\backslash N$ is the direct sum of kernels $b_{1\gamma}(t, X, Y, Z_x, Z_y)$ determined on the invariant subspaces FW^γ. One obviously has

$$b_{1\gamma}(t, X, Y, Z_x, Z_y) = p_{1\gamma}(t, X, Y)e^{2\mathbf{i}\langle Z_\gamma, Z_x - Z_y\rangle}. \tag{145}$$

Because of these straightforward connections, the computations are carried out just for the kernel $d_1(t, X, Y)$. For determining $d_{1\gamma}(t, X, Y)$, defined by a general endomorphism J_γ, one should proceed as follows.

On the maximal subspace where the J_γ is degenerated consider the standard Euclidean heat kernel. Then consider the maximal invariant subspace

on which the J_γ is non-degenerated. Decompose it into orthogonal direct sum of 2-dimensional subspaces which are invariant under the action of J_γ. Such a decomposition can be established by considering a complex coordinate neighborhood $(z_1, \ldots, z_p)$ of the complex structure defined by normalizing J_γ on the non-degenerated subspace. If $\lambda_i \mathbf{i}$ is the eigenvalue on the complex plane corresponding to z_i then consider the kernel $d_{1\lambda_i}(t, X, Y)$ considered in (141). Then the desired kernel, $d_{1\gamma}(t, X, Y)$, is the product of these kernels. Thus the global Wiener-Kac kernel is established in generality.

For two fixed points x and y, this kernel defines the well known complex WK-measure, w_{xy}, on the complete separable metrizable space $\mathcal{P}_{ab}(M)$ of continuous maps $I \to M$ carrying $(0,1) \to (x,y)$. This space is topologized with the topology of uniform convergence [Ee]. It is elementary that the Borel σ-algebra (generated by the open sets) is also generated by the fibred sets $\rho_{\mathbf{t}}^{-1}(B) \subset \mathcal{P}_{xy}(M)$, where $\mathbf{t} = (0 < t_1 < \ldots < t_n < 1$ and $\rho_{\mathbf{t}} : \mathcal{P}_{xy}(M) \to M^n = M \times \ldots \times M$ is the evaluation map $\rho_{\mathbf{t}}(x) = (x(t_1), \ldots, x(t_n))$ and B is a Borel subset of M^n. Measure w_{xy} on a fibred set $\rho_{\mathbf{t}}^{-1}(B)$ is defined by

$$w_{\mathbf{1}xy}(\rho_{\mathbf{t}}^{-1}(B)) = \int_B d_{1\gamma}(t_1, x, m_1) \cdot \tag{146}$$
$$\cdot d_{1\gamma}(t_2 - t_1, m_1, m_2) \ldots d_{1\gamma}(1 - t_n, m_n, y) dm_1 \ldots dm_n.$$

Above the manifold is either compact, or, one considers the problem on the manifold defined by the one-point compactification $M \cup \infty$. Thus the $\mathcal{P}_{xy}$ is a compact topological space. By classical results; such as Riesz' theorem concerning the measure representation of bounded linear functionals on the Banach space of continuous functions defined on a compact metrizable space (the Banach norm is defined by $sup|f|$) and the Stone-Weierstrass theorem asserting that the curves $\rho_{\mathbf{t}_n}^{-1}(x_1, \ldots, x_n)$ corresponding to $\frac{1}{n} < \ldots < \frac{n-1}{n}$ are dense in $\mathcal{P}_{xy}$; this construction determines a complex countably additive regular Borel measure w_{xy} on $\mathcal{P}_{xy}(M)$ satisfying

$$w_{\mathbf{1}xy}(\mathcal{P}_{xy}(M)) = d_{1\gamma}(1, x, y). \tag{147}$$

Next we proceed with the FD-kernel, defined by $\sigma = \mathbf{i}$ in (137), in a similar fashion. In-spite of the infinite multiplicities of the eigenvalues, also this kernel exists which can be established by means of the FD-kernel

$$\frac{-B\mathbf{i}/2}{(2\pi \mathbf{i} t)^{1/2} 2\pi sin(\frac{B}{2}t)} exp(\frac{\mathbf{i}}{2}\{\frac{(x_3 - y_3)^2}{t} \tag{148}$$
$$+ \frac{B}{2}(cot\frac{Bt}{2})[(x_2 - y_2)^2 + (x_1 - y_1)^2] + B(x_1 y_2 - x_2 y_1)\}),$$

established in [FH], formula (3-64), for a rotating charged particle in a constant magnetic field. It is the product of the one-dimensional Euclidean Feynman-Dirac kernel

$$K_0(t,x,y) = (2\pi\mathbf{i}t)^{-1/2} exp(\mathbf{i}\frac{|x-y|^2}{2t}), \tag{149}$$

and of the sought Zeeman-Feynman-Dirac kernel

$$d_{\mathbf{i}}(t,X,Y) = e^{-t\mathbf{i}H_Z}(t,X,Y) = \tag{150}$$
$$(\frac{-B\mathbf{i}/2}{2\pi sin(\frac{1}{2}Bt)})^{k/2} e^{\frac{1}{2}\mathbf{i}\{\frac{B}{2}(cot\frac{Bt}{2})|X-Y|^2 - B\langle X, J(Y)\rangle\}}$$
$$= (\frac{\lambda}{2\pi\mathbf{i}sin(\lambda t)})^{k/2} e^{\mathbf{i}\{\frac{\lambda}{2}cot(\lambda t)|X-Y|^2 - \langle X, J_\gamma(Y)\rangle\}}.$$

Here we suppose that $-J_\gamma^2$ has only one eigenvalue, λ^2. In the general cases this kernel is an appropriate product of kernels of the form (149) and (150).

Also this formula can be established in a self-contained manner by showing that both (138) and the Schrödinger equation $(\partial_t + \mathbf{i}H_{ZX})d_{\mathbf{i}\gamma}(t,X,Y) = 0$ hold. The limit property (138) will be proved later by pointing out that the zonal kernels approach, by the limit $\lim_{t\to+0}$, to the zonal Dirac δ-spreads. The Schrödinger equation can be established by the following computations, what should be carried out also in this case only for $k = 2$.

$$\partial_t(d_{\mathbf{i}\gamma})(t,X,Y) = (-\lambda cot(\lambda t) - \mathbf{i}\frac{\lambda^2|X-Y|^2}{2sin^2(\lambda t)})d_{\mathbf{i}\gamma}(t,X,Y), \tag{151}$$

$$\mathbf{i}(H_Z)_X(d_{\mathbf{i}\gamma})(t,X,Y) = (\lambda cot(\lambda t) + \frac{\lambda^2\mathbf{i}}{2}\{cot^2(\lambda t)|X-Y|^2 \tag{152}$$
$$+|Y|^2 - 2cot(\lambda t)\langle X, J(Y)\rangle - 2cot(\lambda t)\langle J(X), Y\rangle$$
$$-2\langle X, Y\rangle + |X|^2\})d_{\mathbf{i}\gamma}(t,X,Y) = -\partial_t(d_{\mathbf{i}\gamma})(t,X,Y).$$

There arise serious difficulties when one tries to construct the complex Feynman measure on the path-space by the same steps applied for introducing the Wiener-Kac measure. These problems due to the fact that, for fixed t and x, function $d_{\mathbf{i}}(t,x,y)$ (depending on y) is neither of the class L^1 nor L^2 and integral (146) does not exist in general. (Yet, the formally defined Feynman integral is a very useful heuristic tool in perturbation theory. In particular, perturbation expansions can be computed explicitly.) There have been also various attempts made for defining an appropriate Feynman measure. These problems are out of the scope of this paper. Below a non-perturbative theory is offered which is the most important new feature in this article.

For a fixed t the function $d_\sigma(t, X, X)$ is constant, thus the corresponding convolution operators are not of the trace class. Therefore, important functions such as the partition function, zeta function, eta function, and the determinant of the Hamilton operator can not be defined in the usual way. Actually, these non-existing integrals cause the infinities appearing in quantum field theory. In perturbation theory one gets rid off these infinities by "subtracting two infinities" in order to get the desired finite quantity. A typical example is that one adds a suitable potential function V to a non-trace class Hamiltonian H_Z such that the kernel $e^{-tH_Z} - e^{-t(H_Z+V)}$ is of the trace class. Then, one defines the above functions (called relative functions) with respect to this relative kernel. In physics the usual designation for this process is *renormalization*. The most important papers concerning this topic are collected in [Schw].

To implement spectral investigations on non-compact Riemannian manifolds, this tool gained ground also in mathematics. It is called *regularization* which tool includes, besides perturbations, also depicting and removing the divergent terms from functions. This type of investigations started out with [OPS], which article was apparently motivated by papers [A, Po] written in physics. One can consult with [Mü] for finding more informations about the recent developments in this field. So-far this is the major tool for controlling the infinities on both areas. In this article a new non-perturbative approach is offered to these infinities. Namely, the kernels $d_\sigma(t, X, Y)$ will be restricted to the zones and then shown that both zonal kernels $d_\sigma^{(a)}(t, X, Y)$ are trace class kernels. A much more striking discovery is that also the zonal Feynman-Dirac kernels, $d_{\mathbf{i}}^{(a)}(t, X, Y)$, determine a well defined zonal Feynman measure, $w_{\mathbf{i}xy}^{(a)}$, on the path-space.

The most general form of the statements discussed in this section is as follows.

Theorem 5.1 (Global Kernel Theorem) *Infinite function series (137) define smooth kernels both for the Wiener-Kac setting, $\sigma = 1$, and the Dirac-Feynman setting, $\sigma = \mathbf{i}$, where H is the classical Zeeman-Hamilton operator H_Z, or, it is the Zeeman-Hamilton Laplacian H_{Zf}. The kernels corresponding to these operators are distinguished by d_σ and p_σ respectively. These kernels are defined on the X-space $\mathbf{R}^k$. If the kernel is defined on the torus bundle $\Gamma\backslash N$, it is denoted by b_σ. The simple connections among these kernels are described in (144) and (145), thus, it is enough to describe only the classical case.*

Suppose that the Hamilton operator is non-degenerated such that the distinct non-zero parameters $\{\lambda_i\}$, $i = 1, \ldots, r$, are defined on k_i-dimensional

subspaces. Then for the Wiener-Kac kernel we have

$$d_{1\gamma}(t,X,Y) = e^{-tH_Z}(t,X,Y) = \tag{153}$$
$$= \prod (\frac{\lambda_i}{2\pi sinh(\lambda_i t)})^{k_i/2} e^{-\sum \lambda_i (\frac{1}{2} coth(\lambda_i t)|X_i - Y_i|^2 + \mathbf{i}\langle X_i, J(Y_i)\rangle}.$$

This kernel satisfies the Chapman-Kolmogorov identity as well as the limit property (138), however, it is not of the trace class.

The explicit form of the Feynman-Dirac kernel is

$$d_{\mathbf{i}}(t,X,Y) = e^{-t\mathbf{i}H_Z}(t,X,Y) = \tag{154}$$
$$= \prod (\frac{\lambda_i}{2\pi \mathbf{i} sin(\lambda_i t)})^{k_i/2} e^{\mathbf{i}\sum \lambda_i \{\frac{1}{2} cot(\lambda_i t)|X_i - Y_i|^2 - \langle X_i, J(Y_i)\rangle\}}.$$

Since for fixed t and X function $d_{\mathbf{i}}(t,X,Y)$ is neither an L^1- nor an L^2-function of the variable Y, the integral required for the Chapman-Kolmogorov identity is not defined. It is not of the trace class either. Nevertheless, it satisfies the limit property (138).

6 Zonal WK-kernels

Establishing the zonal WK-kernels. Since the zones are spanned by eigenfunctions, the corresponding *zonal Wiener-Kac- and zonal Feynman-Dirac-kernels* can be defined by using only the eigenfunctions belonging to the zone in (137). One can use also the projections $P^{(a)}$ to define these kernels by the integral formula

$$d_\sigma^{(a)}(t,X,Y) = \int\int P^{(a)}(X,U)P^{(a)}(V,Y)d_\sigma(t,U,V)dUdV \tag{155}$$
$$= \int P^{(a)}(X,U)d_\sigma(t,U,Y)dU = \int P^{(a)}(V,Y)d_\sigma(t,X,V)dV.$$

The last two equations follow from the invariance of the zones under the actions of both flows.

The basic tool used in computations below is the well known integral formula:

$$\int_{\mathbf{R}^N} exp(-\frac{1}{2}Z\cdot A\cdot Z + C\cdot Z)dZ = (\frac{(2\pi)^N}{det[A]})^{1/2} exp(\frac{1}{2}C\cdot A^{-1}\cdot C), \tag{156}$$

where A is a complex diagonal matrix having the same entries a, satisfying $Re(a) > 0$, in the diagonal and C is a complex N-vector.

First the zonal kernel $p_1^{(0)}(t,X,Y)$ is established. For the sake of simplicity we assume that $\lambda = 1$. (In the end these formulas will be stated also in the most general form.) By (153) and (156), we have:

$$\int P^{(0)}(X,Z)d_1(t,Z,Y)dZ = \tag{157}$$
$$(\frac{1}{2\pi^2 sinh(t)})^{k/2} e^{-\frac{1}{2}(|X|^2 + coth(t)|Y|^2)} INT,$$

$$INT = \int e^{-\frac{1}{2}Z\cdot A\cdot Z + C\cdot Z} dZ = (\frac{(2\pi)^k}{det[A]})^{1/2} e^{(\frac{1}{2}C\cdot A^{-1}\cdot C)}, \tag{158}$$

where the complex diagonal matrix A has the constant entries

$$a = 1 + coth(t) \tag{159}$$

on the main diagonal and the complex vector C is:

$$C = X - \mathbf{i}J(X) + coth(t)Y - \mathbf{i}J(Y), \tag{160}$$

where J is the complex structure defined by normalizing J_γ. Therefore, by the identities

$$a^{-1} = e^{-t}sinh(t) = \frac{1}{2}(1 - e^{-2t}), \tag{161}$$

$$\frac{1}{2}C\cdot A^{-1}\cdot C = \frac{1}{2}e^{-t}sinh(t)((coth^2(t) - 1)|Y|^2 + \tag{162}$$
$$2(coth(t) - 1)(\langle X, Y\rangle + \mathbf{i}\langle X, J(Y)\rangle)$$

we have

$$d_1^{(0)}(t,X,Y) = \int P^{(0)}(X,Z)d_1(t,Z,Y)dZ = \tag{163}$$

$$= \frac{e^{-\frac{kt}{2}}}{\pi^{\frac{k}{2}}} e^{-\frac{1}{2}(|X|^2 + |Y|^2) + e^{-2t}\langle X, Y + \mathbf{i}J(Y)\rangle}. \tag{164}$$

Note that this kernel satisfies the limit property (143). Unlike the global one, this kernel is of the trace class with trace:

$$Trd_1^{(0)}(t) = \int d_1^{(0)}(t,X,X)dX \tag{165}$$

$$= \int \frac{e^{-\frac{k}{2}t}}{\pi^{\frac{k}{2}}} e^{-(1-e^{-2t})|X|^2} dX = \frac{e^{-\frac{k}{2}t}}{(1 - e^{-2t})^{\frac{k}{2}}}$$

The computations regarding the next kernel,

$$d_1^{(1)}(t,X,Y)=\int P^{(1)}(X,Z)d_1(t,Z,Y)dZ \tag{166}$$
$$=\int L_1^{((k/2)-1)}(|X-Z|^2)P^{(0)}(X,Z)d_1(t,Z,Y)dZ,$$

can be traced back to the above ones by integrating by parts in

$$INT^{(1)}=\int L_1^{((k/2)-1)}(|X-Z|^2)e^{-\frac{1}{2}Z\cdot A\cdot Z+C\cdot Z}dZ.$$

According to this computations we have

$$d_1^{(1)}(t,X,Y)=\Lambda^{(1)}(t,X,Y)d_1^{(0)}(t,X,Y) \tag{167}$$
$$=(L_1^{((k/2)-1)}(|X-Y|^2)+LT_1^{(1)}(t,X,Y))d_1^{(0)}(t,X,Y),$$

where
$$LT_1^{(1)}(t,X,Y))=(1-e^{-2t})lt_1^{(1)}(t,X,Y))= \tag{168}$$
$$(1-e^{-2t})(|X|^2+|Y|^2-1-(1+e^{-2t})\langle X,Y+\mathbf{i}J(Y)\rangle).$$

On a general zone the heat kernel is

$$d_1^{(a)}(t,X,Y)=(L_a^{((k/2)-1)}(|X-Y|^2)+LT_1^{(a)}(t,X,Y))d_1^{(0)}(t,X,Y), \tag{169}$$

thus it is the sum of the *dominant term*

$$D_1^{(a)}=\frac{e^{-\frac{kt}{2}}}{\pi^{\frac{k}{2}}}L_a^{((k/2)-1)}(|X-Y|^2)e^{-\frac{1}{2}(|X|^2+|Y|^2)+e^{-2t}\langle X,Y+\mathbf{i}J(Y)\rangle} \tag{170}$$

and the *long term zonal heat kernel*

$$\tau_1^{(a)}=(1-e^{-2t})T_1^{(a)}(e^{-2t},|X|^2,|Y|^2,\langle X,Y+\mathbf{i}J(Y)\rangle)d_1^{(0)}(t,X,Y), \tag{171}$$

where the $T_1^{(a)}$ is a $2a^{th}$-order polynomial regarding the X,Y variables and a $(2a-1)^{th}$-order polynomial of the e^{-2t} variable. This kernel vanishes for $t=0$, explaining its name. The method of integrating by parts and recursions (62)-(64) yield a recursion formula for the long term kernel $T_1^{(a)}$.

Since a gross zone decomposes into $\binom{a+(k/2)-1}{a}=\binom{a+(k/2)-1}{(k/2)-1}$ number of irreducible zones, each of them is isospectral to the irreducible holomorphic zone, the partition function on a gross zone is

$$\mathcal{Z}^{(a)}(t)=Trd_1^{(a)}(t)=\binom{a+(k/2)-1}{a}\frac{e^{-\frac{k}{2}t}}{(1-e^{-2t})^{\frac{k}{2}}}. \tag{172}$$

On the other hand, by (61), $L_a^{((k/2)-1)}(0) = \binom{a+(k/2)-1}{a} = \binom{a+(k/2)-1}{(k/2)-1}$. Thus the partition function is determined by the dominant heat kernel and the long term heat kernel is in the 0 trace class.

Zonal partition and zeta functions. These formulas allow to introduce the zonal zeta functions by the following standard way:

$$\zeta^{(a)}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathcal{Z}^{(a)}(t) dt = \sum \frac{1}{\mu_i^s}, \tag{173}$$

where $\mu_i > 0$ are the eigenvalues of the Hamilton operator which can be either H_Z or $-\frac{1}{2}\Box_\gamma = H_Z + 2|Z_\gamma|^2 := H_{Zf}$. The infinite series (173) is absolute convergent on the half-plane $Re(s) > 1$ such that the function defined has meromorphic extension onto the whole complex plane.

Interestingly enough, these zeta functions strongly relate to the Riemann, $\zeta_R(s) = \sum \frac{1}{n^s} = \Gamma^{-1}(s) \int_0^\infty t^{s-1}(e^t - 1)^{-1} dt$, and the Hurwitz zeta function, $\zeta_{Hu}(s, x) = \sum \frac{1}{(n+x)^s}$, respectively. (Though Hurwitz defined these functions only for $0 \leq x \leq 1$, we allow arbitrary positive numbers. If x is a natural number, one gets the Hurwitz function by removing the first x terms from the Riemann zeta function.)

In fact, in the original electron case, defined by $k = 2, \lambda = 1$, the multiplicity of eigenvalues on each zone is one and, by (108), one has

$$\zeta_{H_Z}^{(a)}(s) + 2^{-s}\zeta_R(s) = \zeta_R(s)\,,\ \zeta_{H_{Zf}}^{(a)}(s) + 2^{-s}\zeta_{Hu}(s,4) = \zeta_{Hu}(s,4), \tag{174}$$

$$\zeta_R(s) = \frac{2^s}{2^s - 1}\zeta_{H_Z}(s)\,,\ \zeta_{Hu}(s,4) = \frac{2^s}{2^s - 1}\zeta_{H_{Zf}}(s). \tag{175}$$

These equations can be easily established also by the first equation of (173). They provide a quantum physical interpretation for these classical zeta functions.

Also other zonal functions such as zonal determinants, zonal eta- resp. shift-functions can be introduced by the standard formulas originally established on compact Riemannian manifolds. We do not go into these details.

Comparing with the oscillator kernel $e^{-\frac{1}{2}t\mathbf{O}_\gamma}$. Finally, the zonal Wiener-Kac kernels are compared with the global kernel $k_\gamma(t, X, Y)$ defined for the harmonic oscillator $\mathbf{O}_\gamma$ by (137). By (104) this kernel is:

$$k_\gamma(t, X, Y) = \sum e^{\sum_{a=1}^s -(2l_a + k_a)\lambda_{\gamma a} t} \psi_i(X)\overline{\psi}_i(Y), \tag{176}$$

This infinite function series converges to a kernel which is explicitly described by the following Mehler formula:

$$k_\gamma(t,X,Y)=\frac{exp\{\frac{B}{sinh(2Bt)}[-\frac{1}{2}(cosh(2Bt))(|X|^2+|Y|^2)+\langle X,Y\rangle]}{(2\pi sinh(2Bt))^{k/2}} \quad (177)$$

To be more precise, this formula describes the flow with respect to the operator $(1/2)(-\Delta+B|X|^2)$. If there are more constants $(B_1,\ldots,B_s)$ involved, the kernel is the product of kernels belonging to these constants. By this formula one immediately gets that the global Mehler kernel is of the trace class, for any fixed time t.

The infinite function series defining the zonal heat flow is a partial sum of the following infinite function series

$$k_\gamma(2t,X,Y)e^{t\sum(4\upsilon_a+k_a)\lambda_a}, \quad (178)$$

therefore, there is reestablished that the zonal heat kernels are of the trace class on any zone.

Formula (178) provides an opportunity to compare the zonal partition function $\mathcal{Z}^{(a)}(t)$ of the heat flow both with the global and zonal partition function, $\mathcal{Z}_k^{(a)}(t)$, of the oscillator operator $\mathbf{O}_\gamma$. Indeed, one has:

$$0<\mathcal{Z}^{(a)}(t)=\mathcal{Z}_k^{(a)}(t)exp(\sum((4\upsilon_a+k_a)\lambda_a)t) \quad (179)$$
$$<exp(\sum((4\upsilon_a+k_a)\lambda_a)t)/(B(cosh(2Bt)-1))^{k/2},$$

where the last term is computed by the well known partition function of the harmonic oscillator operator.

By summing up we have

Theorem 6.1 (Zonal WK-Flow Theorem) *Suppose that the classical Zeeman-Hamilton operator is non-degenerated, having the non-zero parameters $\{\lambda_i\}$. Then the zonal Wiener-Kac kernels are of the trace class, which can be described, along with their partition functions, by the following explicit formulas.*

$$d_1^{(0)}(t,X,Y)=\prod(\frac{\lambda_i e^{-\lambda_i t}}{\pi})^{\frac{k_i}{2}}e^{\sum\lambda_i(-\frac{1}{2}(|X_i|^2+|Y_i|^2)+e^{-2\lambda_i t}\langle X_i,Y_i+\mathbf{i}J(Y_i)\rangle)}, \quad (180)$$

$$d_1^{(a)}(t,X,Y)=(L_a^{(\frac{k}{2}-1)}(\sum\lambda_i|X_i-Y_i|^2)+LT_1^{(a)}(t,X,Y))d_1^{(0)}(t,X,Y), \quad (181)$$

where $LT_1^{(1)}$ is defined in (168) and for the general terms, $LT_1^{(a)}$, recursion formula can be established. Furthermore,

$$\mathcal{Z}_1^{(a)}(t)=Trd_1^{(a)}(t)=\binom{a+(k/2)-1}{a}\prod\frac{e^{-\frac{k_i\lambda_i t}{2}}}{(1-e^{-2\lambda_i t})^{\frac{k_i}{2}}}=TrD_1^{(a)}(t), \quad (182)$$

where $D_1^{(a)}(t,X,Y) = L_a^{(\frac{k}{2}-1)}(\sum \lambda_i |X_i - Y_i|^2) d_1^{(0)}(t,X,Y)$ *is the dominant zonal kernel. The remaining long term kernel in the WK-kernel vanishes for* $\lim_{t\to 0_+}$ *and is of the 0 trace class. The zonal WK-kernels satisfy the Chapman-Kolmogorov identity along with the limit property (143). (In the text the zonal partition functions and the corresponding zonal zeta functions are described in (163)-(173).)*

In (174) these zeta functions are expressed by the Riemann- resp. Hurwitz zeta functions and vice versa. These formulas provide a quantum physical interpretation for these classical zeta functions.

In (176)-(180) the Wiener-Kac heat kernels are compared with the heat kernel belonging to the harmonic oscillator operator $\mathbf{O}_\gamma$*, which is of the trace class.*

According to (144) and (145), both zonal kernels $p_1^{(a)}$ *and* $b_1^{(a)}$ *can be expressed by* $d_1^{(a)}$*. The traces of these kernels are the multiple of* $Trd_1^{(a)}$ *by the function* $e^{-2t|Z_\gamma|^2}$*.*

The kernels $d_{1\lambda_1 \dots \lambda_{k/2}}^{(a_1 \dots a_{k/2})}$ *on the irreducible zones* $\mathcal{H}^{(a_1 \dots a_{k/2})} \subset \mathcal{H}^{(a)}$ *differ from the above ones just in the Laguerre polynomial term what should be exchanged by* $\prod_{i=1}^{k/2} L_{a_1}^{(0)}$*. The partition function on each irreducible zone is the same as on the holomorphic zone.*

7 Zonal DF-kernels

Integral formula (156) is used also for establishing the zonal Feynman-Dirac kernels. On the holomorphic zone, by assuming $\lambda = 1$, the computations are as follows.

$$\int P^{(0)}(X,Z) d_{\mathbf{i}}(t,Z,Y) dZ = \tag{183}$$
$$(\frac{-\mathbf{i}}{2\pi^2 \sin t})^{k/2} e^{-\frac{1}{2}(|X|^2 - cot(t)|Y|^2 \mathbf{i})} INT,$$

$$INT = \int e^{-\frac{1}{2} Z \cdot A \cdot Z + C \cdot Z} dZ = (\frac{(2\pi)^k}{det[A]})^{1/2} e^{(\frac{1}{2} C \cdot A^{-1} \cdot C)}, \tag{184}$$

where the complex diagonal matrix A has the constant entries

$$a = 1 - \cot t\mathbf{i} = -\mathbf{i} \sin^{-1} t e^{t\mathbf{i}} \tag{185}$$

on the main diagonal and the complex vector C is:

$$C = X - \mathbf{i} J(X) - \mathbf{i}(\cot tY + J(Y)), \tag{186}$$

where J is the complex structure defined by normalizing J_γ. Therefore, by the identities

$$a^{-1} = \sin^2 t(1 + \cot t\mathbf{i}) = \mathbf{i}\sin te^{-t\mathbf{i}}, \tag{187}$$

$$\frac{1}{2}C \cdot A^{-1} \cdot C = \frac{1}{2}(-|Y|^2 + 2\langle X, \cos 2tY + \sin 2tJ(Y)\rangle + \tag{188}$$
$$+(2\langle X, -\sin 2tY + \cos 2t)J(Y)\rangle - \cot t|Y|^2)\mathbf{i}),$$

we have

$$d_{\mathbf{i}}^{(0)}(t, X, Y) = \int P^{(0)}(X, Z)d_{\mathbf{i}}(t, Z, Y)dZ = \tag{189}$$
$$\frac{(\cos t - \mathbf{i}\sin t)^{\frac{k}{2}}}{\pi^{\frac{k}{2}}} e^{-\frac{1}{2}\left(|X|^2+|Y|^2\right)+(\cos 2t-\mathbf{i}\sin 2t)\langle X,Y+\mathbf{i}J(Y)\rangle}$$
$$= \frac{e^{\frac{-kt}{2}\mathbf{i}}}{\pi^{\frac{k}{2}}} e^{-\frac{1}{2}(|X|^2+|Y|^2)+e^{-2t\mathbf{i}}\langle X,Y+\mathbf{i}J(Y)\rangle}.$$

Also this kernel is of the trace class with trace:

$$Trd_{\mathbf{i}}^{(0)}(t) = \int d_{\mathbf{i}}^{(0)}(t, X, X)dX \tag{190}$$
$$= \int \frac{(\cos t - sint\mathbf{i})^{\frac{k}{2}}}{\pi^{\frac{k}{2}}} e^{(\cos 2t-1-\mathbf{i}\sin 2t)|X|^2} dX$$
$$= \frac{(\cos t - \sin t\mathbf{i})^{\frac{k}{2}}}{(1 - \cos 2t + \mathbf{i}\sin 2t)^{\frac{k}{2}}}.$$

The method of integrating by parts applies also to the computations of the higher order zonal DF-kernels. Then one has

$$d_{\mathbf{i}}^{(a)}(t, X, Y) = (L_a^{((k/2)-1)}(|X - Y|^2) + LT_{\mathbf{i}}^{(a)}(t, X, Y))d_{\mathbf{i}}^{(0)}(t, X, Y), \tag{191}$$

where
$$LT_{\mathbf{i}}^{(1)}(t, X, Y)) = (1 - e^{-2t\mathbf{i}})lt_{\mathbf{i}}^{(1)}(t, X, Y)) = \tag{192}$$
$$(1 - e^{-2t\mathbf{i}})(|X|^2 + |Y|^2 - 1 - (1 + e^{-2t\mathbf{i}})\langle X, Y + \mathbf{i}J(Y)\rangle).$$

The long term kernel $LT_{\mathbf{i}}^{(a)}$ on general zones can be described recursively. Also this kernel decomposes into the *dominant term*

$$D_{\mathbf{i}}^{(a)} = \frac{e^{-\frac{kt\mathbf{i}}{2}}}{\pi^{\frac{k}{2}}} L_a^{((k/2)-1)}(|X - Y|^2)e^{-\frac{1}{2}(|X|^2+|Y|^2)+e^{-2t\mathbf{i}}\langle X,Y+\mathbf{i}J(Y)\rangle} \tag{193}$$

and the *long term*

$$\tau_{\mathbf{i}}^{(a)} = (1 - e^{-2t\mathbf{i}})T_1^{(a)}(e^{-2t\mathbf{i}}, |X|^2, |Y|^2, \langle X, Y + \mathbf{i}J(Y)\rangle)d_{\mathbf{i}}^{(0)}(t, X, Y). \tag{194}$$

The same arguments yield that the $T_{\mathbf{i}}^{(a)}$ is a $2a^{th}$-order polynomial regarding the X, Y variables and a $(2a-1)^{th}$-order polynomial of the e^{-2t} variable such that it vanishes for $t = 0$. Furthermore, the long term kernel is of the 0 trace class and the partition function is determined by the dominant term.

By summing up we have

Theorem 7.1 (Zonal Dirac-Feynman Flow Theorem) *Suppose that the classical Zeeman-Hamilton operator is non-degenerated, having the non-zero parameters $\{\lambda_i\}$. Then the gross zonal Dirac-Feynman kernels are of the trace class which, together with their partition functions, can be described by the following explicit formulas.*

$$d_{\mathbf{i}}^{(0)}(t, X, Y) = \prod (\frac{\lambda_i e^{-\lambda_i t\mathbf{i}}}{\pi})^{\frac{k_i}{2}} e^{\sum \lambda_i(-\frac{1}{2}(|X_i|^2+|Y_i|^2)+e^{-2\lambda_i t\mathbf{i}}\langle X_i, Y_i+\mathbf{i}J(Y_i)\rangle)}, \quad (195)$$

$$d_1^{(a)}(t, X, Y) = (L_a^{(\frac{k}{2}-1)}(\sum \lambda_i|X_i - Y_i|^2) + LT_{\mathbf{i}}^{(1)}(t, X, Y))d_1^{(0)}(t, X, Y), \quad (196)$$

where $LT_{\mathbf{i}}^{(1)}$ is described in (192) and a general long term, $LT_{\mathbf{i}}^{(a)}$, can be defined recursively. Furthermore,

$$\mathcal{Z}_{\mathbf{i}}^{(a)}(t) = Trd_{\mathbf{i}}^{(a)}(t) = \binom{a + (k/2) - 1}{a} \prod \frac{e^{-\frac{k_i\lambda_i t\mathbf{i}}{2}}}{(1 - e^{-2\lambda_i t\mathbf{i}})^{\frac{k_i}{2}}} = TrD_{\mathbf{i}}^{(a)}(t), \quad (197)$$

where $D_{\mathbf{i}}^{(a)}(t, X, Y) = L_a^{(\frac{k}{2}-1)}(\sum \lambda_i|X_i - Y_i|^2)d_{\mathbf{i}}^{(0)}(t, X, Y)$ is the dominant kernel. The remaining long term in the zonal DF-kernel is of the 0 trace class.

The zonal DF-kernels are zonal fundamental solutions of the Schrödinger equation: $(\partial_t + \mathbf{i}(H_Z)_X)d_{\mathbf{i}\gamma}^{(a)}(t, X, Y) = 0$, satisfying the Chapman-Kolmogorov identity as well as the limit property (143).

According to (144) and (145), the zonal kernels $p_{\mathbf{i}}^{(a)}$ and $b_{\mathbf{i}}^{(a)}$ can be expressed by $d_{\mathbf{i}}^{(a)}$. The trace of both kernels is the multiple of $Trd_{\mathbf{i}}^{(a)}$ by the function $e^{-2t|Z_\gamma|^2\mathbf{i}}$.

The kernels $d_{\mathbf{i}\lambda_1\ldots\lambda_{k/2}}^{(a_1\ldots a_{k/2})}$ on the irreducible zones $\mathcal{H}^{(a_1\ldots a_{k/2})} \subset \mathcal{H}^{(a)}$ differ from the above ones just in the Laguerre polynomial term what should be exchanged by $\prod_{i=1}^{k/2} L_{a_1}^{(0)}$. The partition function on each irreducible zone is the same as on the holomorphic zone.

8 Zonal path-integrals

Technicalities and constructions on the 0-zone. The existence of zonal Wiener-Kac measure on the path-space follows from the existence of the corresponding global measure. Since such completely additive complex measure does not exist with respect to the global Dirac-Feynman flow, $d_{\mathbf{i}}(t, X, Y)$, it is surprising that the zonal Dirac-Feynman kernels rigorously define complex measures on the path-space. These measures can be established by the standard method used for constructing the Wiener measure. Though it is common, we describe this construction in details.

In the first step consider the one-point compactification $M = \mathbf{v} \cup \{\infty\}$ of the X-space. The *path-space*, on which the measure is to be constructed, consists of curves started out from an arbitrarily chosen point $x \in \mathbf{v}$. Then, for any fixed value $0 < T$, the set $\mathcal{P}_x^T$ of continuous curves $c : [0, T] \to M$ satisfying $c(0) = x$ is a compact metrizable space where the topology is defined by uniform convergence. A continuous curve $c(t)$ is uniquely determined by the rational points corresponding to the parameters $t = rT \in [0, T]$, where $r \in \mathbf{Q}_{[0,1]}$ is an arbitrary rational number on the interval $[0, 1]$. Therefore, this path-space can be identified with the infinite product space

$$\mathcal{P}_x^T = \prod_{t \in \mathbf{Q}_{[0,1]} T} M_t. \tag{198}$$

The Banach space of continuous complex valued functions defined on $\mathcal{P}_x^T$, where the norm is defined by $||F|| = sup|F|$, is denoted by $C(\mathcal{P}_x^T)$. According to the Stone-Weierstrass theorem, this space is generated by continuous functions depending only on finite many of the factors in (198). For a fixed n, the corresponding function space, determined by the rational values $0 < \frac{1}{n} < \ldots < \frac{n-1}{n} < 1$, is denoted by $C_n^\sharp(\mathcal{P}_x^T)$. Then subset $\cup_{n \in \mathbf{N}} C_n^\sharp$ is dense in $\mathcal{P}_x^T$.

The constructions are implemented, first, on the holomorphic zone. By formula (146), the well defined global WK-measure can directly be constructed. An other slightly different construction technique establishes an appropriate *linear functional on* $C(\mathcal{P}_x^T)$, which defines the desired finite complex measure, $w_{\mathbf{1}x}^{T(0)}$, on $\mathcal{P}_x^T$. We follow this idea for constructing the zonal Feynman measures. This functional is defined, first, on subspaces $C_n^\sharp$ by

$$\mathcal{W}_{\mathbf{i}n}^{T(0)}(F) = \tag{199}$$
$$\int d_{\mathbf{i}}^{(0)}(\frac{T}{n}, a, m_1) \ldots d_{\mathbf{i}}^{(0)}(\frac{T}{n}, m_{n-1}, m_n) F(m_1, \ldots, m_n) dm_1 \ldots dm_n$$

$$= \int e^{-\frac{kT}{2}\mathbf{i}} e^{\frac{e^{-2(T/n)\mathbf{i}}-1}{T/n}(T/n)(\langle a,m_1+\mathbf{i}J(m_1)\rangle+\ldots+\langle m_{n-1},m_n+\mathbf{i}J(m_n)\rangle)}$$
$$\cdot\delta^{(0)}(a,m_1)\ldots\delta^{(0)}(m_{n-1},m_n)F(m_1,\ldots,m_n)dm_1\ldots dm_n,$$

where dm is normalized such that $\int |d^{(0)}_{\mathbf{i}}(t,p,m)|^2 dm = ||d^{(0)}_{\mathbf{i}(t,p)}(m)||^2 = 1$ holds for any fixed p and t. By (156) one gets that $dm = \pi^{k/2} dm_E$, where dm_E is the Euclidean volume measure.

At this point of the proof only the first equation is needed. The second one will be used, later, for establishing the Feynman-Kac formula. Let it be also mentioned that functional $\mathcal{W}^{T(0)}_{\mathbf{1}n}(F)$ is similarly defined by means of the zonal Wiener-Kac kernel. In this case all the $\mathbf{i}$'s, but the last ones in formulas $\mathbf{i}J(m_i)$, should be exchanged for 1 in the above formula. The Wiener-Kac functional is uniformly bounded and absolutely continuous which, by the limit $n \to \infty$, extends to the desired absolutely continuous functional $\mathcal{W}^{T(0)}_{\mathbf{1}}(F)$ which then defines the desired measure $w^{T(0)}_{\mathbf{1}x}$.

One can straightforwardly apply these ideas for constructing the zonal Feynman measure. The uniform boundedness of functionals (199) can be established by proving the inequality $|\mathcal{W}^{T(0)}_{\mathbf{i}n}(F)| \leq (2\pi)^{\frac{k}{2}}||F||$, for all n. The proof of this inequality can be carried out by the following formulas calculated by means of (156).

$$\int |d^{(0)}_{\mathbf{i}}(\frac{T}{n},m,p)d^{(0)}_{\mathbf{i}}(\frac{T}{n},p,q)|dp \leq ||d^{(0)}_{\mathbf{i}(T/n,m)}(p)||||d^{(0)}_{\mathbf{i}(T/n,q)}(p)|| = 1, \quad (200)$$

$$\int |d^{(0)}_{\mathbf{i}}(\frac{T}{n},m,p)|||F||dp = \quad (201)$$

$$= \frac{||F||}{\pi^{\frac{k}{2}}} \int e^{-\frac{1}{2}(|m|^2+|p|^2)+cos(\frac{2T}{n})\langle m,p\rangle - sin(\frac{2T}{n})\langle J(m),p\rangle} dp = (2\pi)^{\frac{k}{2}}||F||.$$

If n is even, by applying the first inequality in pairs in (199), one proves that the norms of the functionals $\mathcal{W}^{T(0)}_{\mathbf{i}n}$ are uniformly bounded by 1. If n is odd of the form $n = 2m+1$, then applying the first inequality for the first m pairs of $d_{\mathbf{i}}$'s in (199) and the second one for the last $d_{\mathbf{i}}$, one has the desired inequality. Therefore, by Riesz's and Stone-Weierstrass's theorems, the approximating functionals have a unique continuous extension onto $C(\mathcal{P}^T_x)$.

Note that functional $\mathcal{W}^{T(0)}_{\mathbf{i}n}$ defines approximatig measure $w^{T(0)}_{\mathbf{i}nxy}$ on the approximating path spaces $\mathcal{P}_{nxy} = M \times \ldots \times M$ (n-times product) where the curves have fixed starting point, x, and end point, y. Then we have

$$w^{T(0)}_{\mathbf{i}nxy}(\mathcal{P}_{nxy}) = d^{(0)}_{\mathbf{i}}(T,x,y), \quad (202)$$

what immediately follows from the first equation of (199). Therefore,

$$w^{T(0)}_{\mathbf{i}xy}(\mathcal{P}_{xy}) = \lim_{n\to\infty} w^{T(0)}_{\mathbf{i}nxy}(\mathcal{P}_{nxy}) = d^{(0)}_{\mathbf{i}}(T,x,y). \tag{203}$$

Path-space measure induced by the holomorphic point-spread. Constructions of the zonal measures $\nu^{T(0)}_x$ and $(\nu\overline{\nu})^{T(0)}_x$ on the path-space $\mathcal{P}^T_x$ by means of the point spread kernel

$$\delta^{(0)}(X,Y) = \sum_i \varphi_i(X)\overline{\varphi}_i(Y) = \frac{1}{\pi^{\frac{k}{2}}} e^{-\frac{1}{2}(|X|^2+|Y|^2)+\langle X, Y+\mathbf{i}J(Y)\rangle} \tag{204}$$

and the corresponding density kernel $\delta^{(0)}\overline{\delta}^{(0)}$ on the holomorphic zone can be established by the very same steps described above. These kernels can be used for defining the bounded functionals $\mathcal{N}^{T(0)}_{nxy}$ and $(\mathcal{N}\overline{\mathcal{N}})^{T(0)}_{nxy}$ as well as the corresponding finite measures $\nu^{T(0)}_{nxy}$ and $(\nu\overline{\nu})^{T(0)}_{nxy}$. The functionals turn out to be uniformly bounded and, therefore, defining the sought measures by limit. For the total measure we have

$$\nu^{T(0)}_{xy}(\mathcal{P}_{xy}) = \nu^{T(0)}_{nxy}(\mathcal{P}_{nxy}) = \delta^{(0)}(x,y). \tag{205}$$

The corresponding formula holds for the density measure.

To describe these measures more accurately, we compare them with the well defined Wiener-Kac measure. First note the connection

$$p^{(0)}_1(t,X,Z) = e^{-\frac{kt}{2}} e^{(e^{-2t}-1)\langle X, Z+\mathbf{i}J(Z)\rangle}\delta^{(0)}(X,Z) \tag{206}$$

between $\delta^{(0)}$ and $p^{(0)}_1$. Thus, by taking the limit $n\to\infty$ in the second equation of (199), we have

$$\int_{\mathcal{P}^T_x} f(\omega) dw^{T(0)}_{\mathbf{1}x}(\omega) = \int_{\mathcal{D}^T_x} f(\omega) e^{-\frac{kT}{2}} e^{-2\int_0^T |\omega(\tau)|^2 d\tau} d\nu^{T(0)}_x(\omega). \tag{207}$$

This proves the Radon-Nikodym formula

$$dw^{T(0)}_{\mathbf{1}x}(\omega) = e^{-\frac{kT}{2}} e^{-2\int_0^T |\omega(\tau)|^2 d\tau} d\nu^{T(0)}_x(\omega). \tag{208}$$

The reversed Radon-Nikodym derivative is

$$d\nu^{T(0)}_x(\omega) = e^{\frac{kT}{2}+2\int_0^T |\omega(\tau)|^2 d\tau} dw^{T(0)}_{\mathbf{1}xy}(\omega), \tag{209}$$

describing the considered measure in terms of the Wiener-Kac measure.

The Feynman measure on the 0-zone is constructed by the same steps. In this case the corresponding formulas are

$$p_{\mathbf{i}}^{(0)}(t,X,Z)=e^{-\frac{kt}{2}\mathbf{i}}e^{(e^{-2t\mathbf{i}}-1)\langle X,Z+\mathbf{i}J(Z)\rangle}\delta^{(0)}(X,Z). \tag{210}$$

Thus, for any function $f\in C\mathcal{P}_x^T$, we have

$$\int_{\mathcal{P}_x^T}f(\omega)dw_{\mathbf{i}x}^{T(0)}(\omega)=\int_{\mathcal{D}_x^T}f(\omega)e^{-\frac{kT}{2}\mathbf{i}}e^{-2\mathbf{i}\int_0^T|\omega(\tau)|^2d\tau}d\nu_x^{T(0)}(\omega), \tag{211}$$

$$p_{\mathbf{i}}^{(0)}(T,x,y)=\int_{\mathcal{D}_{xy}^T}e^{(-\frac{kT}{2}-2\int_0^T|\omega(\tau)|^2d\tau)\,\mathbf{i}}d\nu_{xy}^{T(0)}(\omega),$$

$$dw_{\mathbf{i}x}^{T(0)}(\omega)=e^{(-\frac{kT}{2}-2\int_0^T|\omega(\tau)|^2d\tau)\mathbf{i}}d\nu_{xy}^{T(0)}(\omega).$$

By the last equations of (199) resp. (211) we get

$$dw_{\mathbf{i}x}^{T(0)}(\omega)=e^{(\frac{kT}{2}+2\int_0^T|\omega(\tau)|^2d\tau)(1-\mathbf{i})}dw_{\mathbf{1}xy}^{T(0)}(\omega), \tag{212}$$

which is the *Radon-Nikodym derivative of the zonal Feynman measure with respect to the zonal Wiener-Kac measure*. This formula describes the most direct connection between the two zonal measures.

Formulas (207) and (211) can be interpreted as *Feynman-Kac type formulas*. Originally, they stand for the Radon-Nikodym derivative of the well defined Wiener-Kac measure with respect to the Wiener measure concerning the Euclidean Laplacian Δ_X. On the zonal setting this idea can not be carried out because the zones are not invariant with respect to the Euclidean Laplacian and the zonal decomposition with respect to this Laplacian is not defined. In the above version, measure $d\nu_x^{T(0)}$ substitutes the Wiener measure.

The zonal Feynman measure of the set of curves connecting x and y is

$$w_{\mathbf{i}xy}^{T(0)}(\mathcal{P}_{xy}^T(M))=d_{\mathbf{i}}^{(0)}(T,x,y). \tag{213}$$

This function is called zonal (in this case, "holomorphic") *probability amplitude* and the function defined by

$$\rho_{xy}^{T(0)}=d_{\mathbf{i}}^{(0)}(T,x,y)\overline{d}_{\mathbf{i}}^{(0)}(T,x,y) \tag{214}$$

is called zonal (holomorphic) *probability density*. For a Borel set B on the X-space, integral $\int_B\rho_{x,y}^{(0)}dy$ (where dy is the normalized density described in (199)) measures the probability of that that the point spread about x can be caught, at the time T, among the point-spreads spread-ed about the

points of B, meaning that $d^{(0)}_{\mathbf{i}}(T,x,y)$ is in the function space spanned by functions $d^{(0)}_{\mathbf{i}}(T,b,y), b \in B$. For $B = \mathbf{R}^k$, this probability is 1.

Probability density can be defined also for any zonal function $\psi^{(0)}(t,X)$ satisfying the Schrödinger equation. By the convolution formula

$$\psi^{(0)}(t,X) = d^{(0)}_{\mathbf{i}}(t,X,Z) *_Z \psi(0,Z), \tag{215}$$

such a function is uniquely determined by the function defined for $t = 0$. If the initial function $\psi^{(0)}(0,X)$ is normalized, so is $\psi^{(0)}(t,X)$, for any t. This statement readily follows from

$$\frac{1}{\mathbf{i}}\frac{\partial d^{(0)}_{\mathbf{i}}}{\partial t} = -(H_Z)_X d^{(0)}_{\mathbf{i}} \quad , \quad \frac{1}{\mathbf{i}}\frac{\partial \overline{d}^{(0)}_{\mathbf{i}}}{\partial t} = \overline{(H_Z)_X d^{(0)}_{\mathbf{i}}} \tag{216}$$

and from $\int \overline{\Box f} g = \int \overline{f} \Box g$, for all $f, g \in \mathcal{H}^{(0)}$. Indeed, by them we have $\partial_t \int \psi\overline{\psi} = 0$. This statement is known as *conservation of probability under the action of the DF-flow.*

The probability density is defined by $\rho^T_\psi = \psi\overline{\psi}$. In (210) the Dirac delta spread about an arbitrary point x is the initial function by which the probability amplitude is defined. In general the initial function defines a starting continuum-spread (zonal object) and $\int_B \psi(T,X)\overline{\psi}(T,X)dX$ measures the probability that the zonal object can be caught on the Borel set B, at the time T.

The actual mathematical theorem exhibited in the above theorem can be formulated such that that *the holomorphic Feynman-Dirac flow defines a unitary semi-group*, $U^{(0)}_t$, on the holomorphic zone.

Constructions on general zones. Since the conservation of probability holds on each zone, formula (200) extends from the 0-zone to any zone. Also (201) extends to each zone. In fact, functions $d^{(a)}_{\mathbf{i}}(\frac{T}{n}, m, p)$ depending on p are absolute integrable for any fixed n and m, furthermore,

$$\lim_{n \to infty} \int |d^{(a)}_{\mathbf{i}}(\frac{T}{n}, m, p)|dp = \int |\delta^{(a)}(m,p)|dp.$$

From this convergence an appropriate upper bound for the generalized integral (201) follows, proving that the zonal Feynman measures are well defined on each zone. Thus we have

Theorem 8.1 (Zonal Path Integral Theorem) *Let H_Z be a non-degenerated Zeeman-Hamilton operator with parameters $\lambda_1, \ldots, \lambda_r$. Then, for each zone, both zonal kernels $p^{(a)}_\sigma, \sigma = 1, \mathbf{i}$, generate continuous measures $dw^{T(a)}_{\sigma xy}$*

on the path-space $\mathcal{P}_{xy}^T$ such that the corresponding measures of the whole path-space are $p_\sigma^{(a)}(T,x,y)$. They are called zonal Wiener-Kac and Feynman measures respectively.

Functions $\pi^{k/2}d_{\mathbf{i}}^{(a)}(T,x,y)\overline{d}_{\mathbf{i}}^{(a)}(T,x,y)$ resp. $\psi^{(a)}\overline{\psi}^{(a)}$, where $\psi^{(a)}(t,X)$ is a zonal function satisfying the Schrödinger equation and $||\psi^{(a)}(0,X)|| = 1$, are called zonal probabilistic density functions. For any T, the norm of these functions is 1. This phenomena is called conservation of probability. Since $\psi^{(a)}(t,X) = \int d_{\mathbf{i}}(t,Z)\psi^{(a)}(0,Z)$, the mathematical meaning of this statement is that the Feynman-Dirac flow generates, on each zone, a semi-group $U_t^{(a)}$ of unitary transformations.

A zonal measure, $d\nu_{xy}^{T(a)}(\omega)$, is determined on $\mathcal{P}_{xy}^T$ also by the zonal Dirac spread $\delta_{\lambda_1\dots\lambda_r}^{(a)}$. On the holomorphic zone the Radon-Nikodym derivative of the measure $dw_{\mathbf{i}x}^{T(0)}$ with respect to the $d\nu_x^{T(0)}$ is described in the following Feynman-Kac type formulas.

$$\int_{\mathcal{P}_x^T} f(\omega)dw_{\mathbf{i}x}^{T(0)} = \int_{\mathcal{P}_x^T} f(\omega)e^{\sum\lambda_i(-\frac{k_iT}{2}\mathbf{i}-2\mathbf{i}\int_0^T|\omega_i(\tau)|^2d\tau)}d\nu_x^{T(0)}, \qquad (217)$$

$$d_{\mathbf{i}}^{(0)}(T,x,y) = \int_{\mathcal{P}_{xy}^T} e^{\sum\lambda_i(-\frac{k_iT}{2}-2\int_0^T|\omega_i(\tau)|^2d\tau)\,\mathbf{i}}d\nu_{xy}^{T(0)}(\omega).$$

The Radon-Nikodym derivative of $\nu_x^{T(0)}$ with respect to $w_{\mathbf{1}x}^{T(0)}$ is established in (209).

On the holomorphic zone, the Feynman-Kac type formula regarding the WK-measure is:

$$d_1^{(0)}(T,x,y) = \int_{\mathcal{P}_{xy}^T} e^{\sum\lambda_i(-\frac{k_iT}{2}-2\int_0^T|\omega_i(\tau)|^2d\tau)}d\nu_{xy}^{T(0)}(\omega). \qquad (218)$$

These two Feynman-Kac type formulas establish the Radon-Nikodym derivative

$$dw_{\mathbf{i}xy}^{T(0)}(\omega) = e^{\sum\lambda_i(\frac{k_iT}{2}+2\int_0^T|\omega_i(\tau)|^2d\tau)(1-\mathbf{i})}dw_{\mathbf{1}xy}^{T(0)}(\omega) \qquad (219)$$

of the holomorphic Feynman measure with respect to the holomorphic Wiener-Kac measure, providing the most direct connection between these two holomorphic measures.

These zonal measures can be constructed also by means of the zonal kernels $p_\sigma^{(a)}$ and $b_\sigma^{(a)}$. The two modifications should be implemented regarding the Feynman-Kac type formulas are the following ones: (1) The above functions are multiplied by $e^{-2t|Z_\gamma|^2}$. (2) The Dirac δ-spread on the torus bundle

regarding the function space FW^γ is defined such that $\delta_\gamma^{(a)}$ is multiplied by $e^{2\mathbf{i}\langle Z_\gamma, Z_x - Z_y\rangle}$ and measure $d\nu$ regards this δ-spread. Then the Feynman-Kac formulas for these two zonal kernels have the same form.

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